On Optimal LTI Approximation of Nonlinear Systems
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Abstract—Optimal linear time-invariant (LTI) approximation of discrete-time nonlinear systems is studied using a quasi-stationary signal description. The optimality of the LTI approximation is quantified as a least average squared error condition, reminiscent of recent studies on the performance of least squares type identification methods for LTI models of nonlinear systems.

Index Terms—Approximation, least squares, linear models, nonlinear systems, quasi-stationarity.

I. INTRODUCTION
Linear time-invariant (LTI) models are used very commonly in applications of control theory, although real plants are typically at least mildly nonlinear. In this note, we establish an optimal LTI approximation result for single-input—single-output (SISO) nonlinear, time-varying, discrete-time systems using a quasi-stationary signal setup.

Recently there has been considerable interest in a theory of optimal LTI modeling for nonlinear systems, see [11], [6], and [7]. This is a rich research area with many open problems. One motivation for such studies comes from the importance to understand the impact of nonlinear distortions on the performance of standard LTI model estimation methods [10], [12], [6], [2]. Another motivation comes from the need to understand the design of LTI controllers based on LTI approximations of nonlinear systems, see for example [16], [11] and [7].

There are many ways to define optimality of an LTI model of a nonlinear system. Best LTI approximation of nonlinear systems so as to minimize an induced system gain error criterion is studied in [7]. In [11], an average squared error criterion is used, for continuous-time systems, in a signal setup allowing generalized harmonic analysis [14], [15]. In the discrete-time case this setup corresponds to the signal space of quasi-stationary signals in the terminology of Ljung [5], [6]. Such signal spaces have also been used in generalizations of robust $H_\infty$ control to persistent signal spaces using the terminology bounded power signal spaces [17], [8]. The associated root-mean-square (RMS) signal size measure has been discussed as a controller design performance criterion for example in [1], for LTI systems.

We shall use a (purely deterministic) quasi-stationary signal setup in the present note. A technical problem with the space of quasi-stationary signals is that it is not a linear space as the sum of two quasi-stationary signals need not be a quasi-stationary signal. It can be seen that the associated $H_\infty$ and $l_1$ control are generalized in [8] and [9] to large linear spaces of persistent signals so that quasi-stationary signals are included as a subset.

II. OPTIMAL APPROXIMATION PROBLEM
We begin by introducing some notation.

Let $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ denote the nonnegative integers, the integers, and the reals, respectively. The space $s(\mathbb{N})$ is the linear space of all real sequences $x = \{x(k) \in \mathbb{R}\}_{k \in \mathbb{N}}$ over $\mathbb{N}$. The linear normed space $\ell_{\infty}(\mathbb{N})$ is the space of all real sequences $x = \{x(k) \in \mathbb{R}\}_{k \in \mathbb{N}}$ such that

$$\|x\|_{\infty} = \sup_{k \in \mathbb{N}} |x(k)| < \infty.$$ 

Furthermore, the linear normed space $\ell_1(\mathbb{Z})$ is the space of all real sequences $x = \{x(k) \in \mathbb{R}\}_{k \in \mathbb{Z}}$ such that

$$\|x\|_1 = \sum_{k \in \mathbb{Z}} |x(k)| < \infty.$$ 

Let $x \in s(\mathbb{N})$, i.e., let $x$ be a signal, and introduce the autocovariance of the signal $x$ as

$$R_{xx}(k) = \lim_{n \to \infty} \frac{1}{n} \sum_{t=0}^{n-1} x(t)x(t+k), k \in \mathbb{Z}.$$ 

(Here, we put $x(t) = 0$ for $t < 0$.) It holds that $R_{xx}(-k) = R_{xx}(k)$ if $R_{xx}(k)$ exists. The autocovariance sequence $R_{xx} = \{R_{xx}(k) \in \mathbb{R}\}_{k \in \mathbb{Z}}$ need not exist for a general signal $x$.

We use the terminology of [6] and we say that a signal $x$ is quasi-stationary if $x \in \ell_{\infty}(\mathbb{N})$ and $x$ possesses an autocovariance sequence $R_{xx}$. Furthermore, we say that the two real sequences $v$ and $x$ possess a crosscovariance sequence if

$$R_{vx}(k) = \lim_{n \to \infty} \frac{1}{n} \sum_{t=0}^{n-1} v(t)x(t+k)$$

exists for any $k \in \mathbb{Z}$. We write $R_{vx} = \{R_{vx}(k) \in \mathbb{R}\}_{k \in \mathbb{Z}}$. Let $G : D(G); s(\mathbb{N}) \to s(\mathbb{N})$ denote a nonlinear system, i.e., a nonlinear input–output operator

$$y = Gu$$

where $y$ is the output and $u \in D(G); s(\mathbb{N})$ is the input. Here, $D(G); s(\mathbb{N})$ denotes the domain of definition of $G$ in $s(\mathbb{N})$, i.e., the set of all $u \in s(\mathbb{N})$ such that $y = Gu \in s(\mathbb{N})$.

An optimal LTI approximation problem for $G$ is now introduced as follows. Let $u \in D(G); s(\mathbb{N})$. Consider

$$\inf_{F} \lim_{n \to \infty} \frac{1}{n} \sum_{t=0}^{n-1} [(Gu)(t) - (Fu)(t)]^2$$

where the infimum is taken over all linear time-invariant (LTI) systems $F$ having an absolutely summable kernel $f = \{f(k)\}_{k \in \mathbb{Z}} \in \ell_1(\mathbb{Z})$. Here

$$(Fu)(t) = \sum_{k \in \mathbb{Z}} f(k)u(t-k), t \in \mathbb{N}.$$ 

(We put $u(t) = 0$ for $t < 0$.) Note that we allow for noncausal $G$ and $F$.

It is clear that additional conditions are required for the limit in (1) to exist. Such conditions are given in the main result of this note. To state this result, let us introduce the (infinite Toeplitz) matrix

$$[S_n]_{kl} = R_{uu}(k\cdot l-I), k,l \in \mathbb{Z}$$

when $u$ is quasi-stationary. We shall also interpret $S_n$ as a linear operator from $\ell_1(\mathbb{Z})$ into $\ell_1(\mathbb{Z})$ with domain of definition $D(S_n; \ell_1(\mathbb{Z}))$.

Theorem 2.1: Let the nonlinear system $G : D(G); s(\mathbb{N}) \to s(\mathbb{N})$, with a nonempty domain of definition, be given. Let $u \in D(G); s(\mathbb{N})$ be quasi-stationary. Let $D(S_n; \ell_1(\mathbb{Z})) = \ell_1(\mathbb{Z})$ and let $S_n : \ell_1(\mathbb{Z}) \to \ell_1(\mathbb{Z})$ have a bounded inverse $S_n^{-1} : \ell_1(\mathbb{Z}) \to \ell_1(\mathbb{Z})$. Let $R_{uu}(0)$ exist and let $R_{uu} \in \ell_1(\mathbb{Z})$. Then, the optimal approximation problem (1)

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has a unique solution $F^*$, called the best LTI approximation of $G$, with kernel $f^* \in \ell_1(\mathbb{Z})$ given by

$$f^* = S_n^{-1} R_{uv}$$

and the infimum (minimum) in (1) is

$$\inf_F \lim_{n \to \infty} \frac{1}{n} \sum_{t=0}^{n-1} |G(u)(t) - (F(u))(t)|^2 = R_{uv}(0) - R_{uv}^2 S_n^{-1} R_{uv}.$$ 

Standard conventions for matrix and vector operations are used here (a summary of these can be found in the proof to follow). Furthermore, clearly $S_n^{-1} S_n x = S_n^{-1} x = x$ for any $x \in \ell_1(\mathbb{Z})$, and a bounded inverse $S_n^{-1}: \ell_1(\mathbb{Z}) \to \ell_1(\mathbb{Z})$ means that the (induced) operator norm of the linear operator $S_n^{-1}$ is finite, i.e.,

$$\|S_n^{-1}\|_1 \equiv \sup_{\|x\|_1 = 1} \|S_n^{-1} x\|_1 < \infty.$$ 

Note that the optimal LTI approximation is provided by a correlation estimate [3], [11].

**Proof:** Note that the assumption $D(S_n; \ell_1(\mathbb{Z})) = \ell_1(\mathbb{Z})$ holds if and only if (iff) $R_{uv} \in \ell_1(\mathbb{Z})$, i.e., iff $R_{uv}$ is an absolutely summable real sequence. We verify this as follows. Denote $z = S_n x$, or

$$z(k) = \sum_{j \in \mathbb{Z}} [S_n]_{kj} x(j), \quad k \in \mathbb{Z}$$

here, $x \in \ell_1(\mathbb{Z})$. Put $x(0) = 1$ and $x(j) = 0$, otherwise. Then $z(k) = [S_n]_{10} = R_{uv}(k), \quad k \in \mathbb{Z}$. Hence, $z \in \ell_1(\mathbb{Z})$ only if $R_{uv} \in \ell_1(\mathbb{Z})$. This proves the necessity part of our claim. Now

$$|z(k)| \leq \sum_{j \in \mathbb{Z}} |R_{uv}(k - j)||x(j)|, \quad k \in \mathbb{Z}$$

and hence

$$\sum_{k \in \mathbb{Z}} |z(k)| \leq \sum_{k, j \in \mathbb{Z}} |R_{uv}(k - j)||x(j)| \leq \sum_{k \in \mathbb{Z}} |R_{uv}(k)| \sum_{j \in \mathbb{Z}} |x(j)|.$$ 

So, $R_{uv} \in \ell_1(\mathbb{Z})$ is a sufficient condition for $S_n x \in \ell_1(\mathbb{Z})$ to hold for any $x \in \ell_1(\mathbb{Z})$. The verification is complete.

Denote $y = Gu$. We need to study the partial sums

$$s_n = \frac{1}{n} \sum_{t=0}^{n-1} (y(t) - (Fu)(t))^2, \quad n \geq 1.$$ 

(Note that $u(0) = 0$ or $t < 0$.) We shall in the sequel denote $s_n(f) = s_n$ to emphasize the dependence of $s_n$ on $f$.

Now

$$s_n(f) = \frac{1}{n} \sum_{t=0}^{n-1} y(t)^2$$

$$- \frac{1}{n} \sum_{t=0}^{n-1} y(t)(Fu)(t) + \frac{1}{n} \sum_{t=0}^{n-1} (Fu)(t)^2.$$ 

(2)

Clearly, the first sum in (2) satisfies

$$\lim_{n \to \infty} \frac{1}{n} \sum_{t=0}^{n-1} y(t)^2 = R_{uv}(0)$$

as $R_{uv}(0)$ exists by assumption.

We proceed to study the sum $\frac{1}{n} \sum_{t=0}^{n-1} y(t)(Fu)(t)$ when $n \to \infty$. We can interchange the order of summation as follows:

$$\frac{1}{n} \sum_{t=0}^{n-1} y(t)(Fu)(t) = \frac{1}{n} \sum_{t=0}^{n-1} y(t) \sum_{k=0}^{n-1} f(k) u(t - k)$$

$$= \sum_{k \in \mathbb{Z}} f(k) \left[ \frac{1}{n} \sum_{t=0}^{n-1} y(t) u(t - k) \right]$$

because $u \in \ell_{\infty}(\mathbb{N}), R_{uv}(0)$ exists, and $f \in \ell_1(\mathbb{Z})$ by assumption. To verify this we compute, letting $p$ and $q$ be any positive integers

$$\frac{1}{n} \sum_{t=0}^{n-1} y(t)(Fu)(t) - \sum_{\ell \leq q \leq p} f(k) \left[ \frac{1}{n} \sum_{t=0}^{n-1} y(t) u(t - k) \right]$$

$$\leq \left( \frac{1}{n} \sum_{t=0}^{n-1} y(t)^2 \right)^{1/2} \sum_{\ell \leq q \leq p} f(k) \|u\|_{\infty}.$$ 

However, the square root expression is bounded in $n$ as $R_{uv}(0)$ exists, and so as $u \in \ell_{\infty}(\mathbb{N})$ and $\sum_{k \leq q \leq p} f(k) \to 0$ when $p$ and $q$ tend to infinity, the summation order can indeed be interchanged. Note that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{t=0}^{n-1} y(t) u(t - k) = R_{uv}(k)$$

as $R_{uv}(k)$ exists for any $k \in \mathbb{Z}$ by assumption. Hence, the second sum in (2) satisfies

$$\lim_{n \to \infty} \frac{1}{n} \sum_{t=0}^{n-1} y(t)(Fu)(t) = \sum_{k \in \mathbb{Z}} f(k) R_{uv}(k) \in \mathbb{R}$$

as $f \in \ell_1(\mathbb{Z})$ and $R_{uv} \in \ell_1(\mathbb{Z})$.

Finally, the third sum in (2) gives

$$\frac{1}{n} \sum_{t=0}^{n-1} (Fu)(t)^2 = \frac{1}{n} \sum_{t=0}^{n-1} \left[ \sum_{k \in \mathbb{Z}} f(k) u(t - k) \right] \left[ \sum_{\ell \in \mathbb{Z}} f(\ell) u(t - \ell) \right]$$

$$= \sum_{k, \ell \in \mathbb{Z}} f(k) f(\ell) \left[ \frac{1}{n} \sum_{t=0}^{n-1} u(t - k) u(t - \ell) \right]$$

as $f \in \ell_1(\mathbb{Z})$ and $u \in \ell_{\infty}(\mathbb{N})$. However

$$\lim_{n \to \infty} \frac{1}{n} \sum_{t=0}^{n-1} u(t - k) u(t - \ell) = R_{uv}(k - \ell)$$

as $u \in \ell_{\infty}(\mathbb{N})$ and $R_{uv}(k - \ell)$ exists for any $k, \ell \in \mathbb{Z}$. Therefore

$$\lim_{n \to \infty} \frac{1}{n} \sum_{t=0}^{n-1} (Fu)(t)^2 = \sum_{k, \ell \in \mathbb{Z}} f(k) f(\ell) R_{uv}(k - \ell) \in \mathbb{R}$$

as $f \in \ell_1(\mathbb{Z})$ and $R_{uv} \in \ell_1(\mathbb{Z})$.

Hence

$$L(f) \equiv \lim_{n \to \infty} s_n(f) = R_{uv}(0) - 2 \sum_{k \in \mathbb{Z}} f(k) R_{uv}(k)$$

$$+ \sum_{k, \ell \in \mathbb{Z}} f(k) f(\ell) R_{uv}(k - \ell).$$
Thus, the approximation error (denoted here as $L(f)$) in (1) is well-defined for any $f \in \ell_1(\mathbb{Z})$, i.e., the domain of definition of $L$ is the whole space $\ell_1(\mathbb{Z})$. We use the inner product notation
$$v^T x \equiv \sum_{i \in \mathbb{Z}} v(k)x(k)$$
and the quadratic form notation
$$x^T Qx = \sum_{i, \ell \in \mathbb{Z}} x(k)Q_i \ell v(\ell)$$
where $x$ and $v$ are infinite-dimensional vectors and $Q$ is an infinite-sized matrix. With this notation, we can write
$$L(f) = R_{uv}(0) - 2f^T R_{uv} + f^T S_n f.$$  
(3)

We proceed by making the substitution $f = S_n^{-1} R_{uv} + x$. Here, $S_n^{-1} R_{uv}$ (the matrix product is defined in the usual manner) is an infinite-dimensional vector in $\ell_1(\mathbb{Z})$ by the boundedness of $S_n^{-1}$ and as $R_{uv} \in \ell_1(\mathbb{Z})$, and so $x \in \ell_1(\mathbb{Z})$. Then
$$K(x) \equiv L(S_n^{-1} R_{uv} + x) = R_{uv}(0) - 2(S_n^{-1} R_{uv} + x)^T R_{uv} + (S_n^{-1} R_{uv} + x)^T S_n (S_n^{-1} R_{uv} + x)$$
and so the minimization of $L(f)$ over $f \in \ell_1(\mathbb{Z})$ is equivalent to the minimization of $K(x)$ over $x \in \ell_1(\mathbb{Z})$. Note that $S_n$ and so $S_n^{-1}$ are symmetric matrices as $R_{uv}(k) = R_{uv}(-k)$ for any $k \in \mathbb{Z}$. Thus, $K(x)$ can be written as
$$K(x) = R_{uv}(0) - R_{uv}^T S_n^{-1} R_{uv} + x^T S_n x.$$  
(4)

However, clearly only the last term in this expression for $K(x)$ depends on $x$ and as this term is nonnegative for any $x$ (this is easiest to see by observing that the analogous term $f^T S_n f$ in (3) is nonnegative for it has been obtained as the limit of a sequence of nonnegative quantities), it follows that
$$\inf_{x \in \ell_1(\mathbb{Z})} L(f) = \inf_{x \in \ell_1(\mathbb{Z})} K(x) = R_{uv}(0) - R_{uv}^T S_n^{-1} R_{uv}$$
and a minimizer of $K(x)$ is $x = 0$, and so $f^* \equiv S_n^{-1} R_{uv}$ is a minimizer of $L(f)$ in $\ell_1(\mathbb{Z})$.

It remains to establish uniqueness of the minimizer. Let $S^{(m)}$ denote the upper left corner submatrix of $S_n$ of size $m \times m$, where $m$ is a positive integer. $S^{(m)}$ must be a positive–definite matrix as $S_n$ is invertible by assumption. Let $\lambda_i^{(m)} > 0$, $i = 1, \ldots, m$, denote the eigenvalues of $S^{(m)}$. Clearly
$$\inf_{m \geq 1} \min_{1 \leq i \leq m} \lambda_i^{(m)} > 0$$
as $S_n^{-1}$ is bounded by assumption. Therefore, as $R_{uv} \in \ell_1(\mathbb{Z})$, it holds for any nonzero $x \in \ell_1(\mathbb{Z})$ that
$$x^T S_n x > 0.$$  

It then follows by (4) that $x = 0$ is the unique minimizer of $K(x)$. This completes the proof.

Note that the assumption $R_{uv}(0) \in \mathbb{R}$ is, in general, milder than the assumption that $y$ is quasi-stationary, see [8, p. 107]. Observe also that $u$ being quasi-stationary does not, in general, imply that $y$ is quasi-stationary nor that $R_{uv}(0)$ exists, see [8, pp. 104–105]. It should also be noted that as we allow noncausal LTI approximations $F$, it is not possible to relax the condition that the kernel $f$, of $F$, is absolutely summable. Clearly, this result can be readily modified to the case that $F$ is restricted to be causal.

It is well known, see, for example, [14] and [8], that using the spectrum, $H_u$, of $R_{uv}$, we can write
$$R_{uv}(k) = \int_{-\pi}^{\pi} e^{i\omega t} dH_u(\omega).$$
($H_u(\omega)$ is a nondecreasing, square integrable function on $[-\pi, \pi]$).

Furthermore, it is common to introduce the spectral density function $h_u(\omega)$ of $R_{uv}$ as
$$h_u(\omega) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} R_{uv}(k) e^{-i\omega k}, -\pi \leq \omega \leq \pi.$$  

Similarly, introduce the spectrum $H_y$ and the spectral density $h_y$ of $R_{yy}$, when $R_{yy}$ exists. Finally, introduce the cross-spectral density
$$h_{yu}(\omega) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} R_{uv}(k) e^{-i\omega k}, -\pi \leq \omega \leq \pi.$$  

We now observe that the conditions on $R_{uv}$ ($S_n$ and $S_n^{-1}$) in the aforementioned result are equivalent to $h_u(\omega)$ having absolutely summable Fourier coefficients and no zero in $[-\pi, \pi]$ due to Wiener’s theorem [4]. The following corollary to Theorem 2.1 follows then by a straightforward computation.

**Corollary 2.1:** The unique minimizer $f^*$ in Theorem 2.1 is determined by
$$f^*(\epsilon \omega) = \frac{h_{yu}(\omega)}{h_y(\omega)},$$
where $f^*(\epsilon \omega)$ denotes the frequency response of $f^* \in \ell_1(\mathbb{Z})$, i.e.,
$$f^*(\epsilon \omega) = \sum_{k \in \mathbb{Z}} f^*(k) e^{-i\omega k}, -\pi \leq \omega \leq \pi.$$  

Note also that as $1/ h_y$ has absolutely summable Fourier coefficients only if $h_y(\omega)$ has no zero in $[-\pi, \pi]$, it follows that the conditions on $R_{uv}$ in Theorem 2.1 are as weak as possible.

Finally, note that even if $y$ were quasi-stationary, it is not possible, in general, to express the infimal value of the approximation criterion in Theorem 2.1 only with spectral densities as
$$R_{uv}(0) = H_y(\pi) - H_y(-\pi) \geq \int_{-\pi}^{\pi} h_y(\omega) d\omega$$
and, in general, strict inequality applies here ($H_y$ may contain a singular part). However, strict equality applies above if $R_{uv} \in \ell_1(\mathbb{Z})$.

We end this section with a signal quasi-stationarity result. Let $(x)$ denote the fractional part of $x \in [0, \infty)$ (so that $0 \leq (x) < 1$ and $x = (x) \in \mathbb{N}$).

**Theorem 2.2:** Let $P$ be a real function on $[0, \infty)$ satisfying $P(x) = P(P(x))$ for any $x \in [0, \infty)$ and let $P$ be continuous, but not identically zero, on $[0, 1]$. Consider the sequence $u = \{u(t)\}_{t \geq 0}$, where $u(t) = P(\gamma t), t \geq 0$, and $\gamma$ is a positive irrational number. Then, $u$ is quasi-stationary, but $R_{uv}$ is not absolutely summable.

Note that $P$ is periodic with period length 1 (any periodic function can be made to have period length 1 by a scaling of variables).

**Proof:** Take $k \in \mathbb{N}$ We compute
$$\frac{1}{n} \sum_{t=0}^{n-1} P(\gamma t) P(\gamma (t + k)) = \frac{1}{n} \sum_{t=0}^{n-1} P(\gamma t) P(\gamma (t) + \gamma (k))$$
and so as by Weyl’s theorem [13, pp. 11–12] $\{ \gamma (t) \}_{t \geq 0}$ is equidistributed in $(0, 1)$, it follows by the continuity of $P$ on $[0, 1]$ that
$$R_{uv}(k) = \int_{0}^{1} P(x) P(x + \gamma k) dx.$$
Clearly, $R_n(-k) = R_n(k)$ for any $k \in \mathbb{N}$ (by convention we put $u(t) = 0$ for $t < 0$). Hence $R_n$ exists and as $\|v\|_\infty = \max_{t \in [0,1]} |P(t)| < \infty$ by the continuity of $P$, it is seen that $v$ is quasi-stationary. Now, as $P$ is not identical to zero and continuous, it follows that $R_n(0) > 0$. As $\{(k\gamma)\}_{k \geq 0}$ is equidistributed in $(0,1)$, it follows that there exists a strictly monotonically increasing sequence $\{k_i\}_{i \geq 0}$ of positive integers such that
\[
|R_n(k_i)| \geq \frac{R_n(0)}{2}, \quad i \geq 1
\]
by the continuity of $P$. Hence, indeed $R_n$ is not absolutely summable. This completes the proof.

With this result it is seen that there exist rather irregular looking quasi-stationary input sequences $u$, but which do not satisfy the absolute summability assumption on $R_n$ in Theorem 2.1. For such sequences the error criterion in (1) is not defined for an arbitrary stable LTI model with absolutely summable unit impulse response. Furthermore, the condition $D(S_t; t_1(Z)) = t_1(Z)$ can not then hold. This points to the fact that the assumptions used in Theorem 2.1 are quite nontrivial.

III. APPLICATION

In this section, an application of Theorem 2.1 is given to a particular class of nonlinear systems.

Consider the nonlinear system $y = Gu$ given in state-space form as
\[
\begin{align*}
x(t+1) &= Ax(t) + B(u(t))u(t), \quad x(0) = 0 \\
y(t) &= Cx(t)
\end{align*}
\]
where $x$ is the state vector of dimension $m$, $A$ is a square matrix with all eigenvalues strictly inside the unit circle, and $C$ is a row vector. Finally, $B : \mathbb{R} \to \mathbb{R}^m$ satisfies $B(v) = B(v)$ for any $v \in \mathbb{R}$ (Here $\mathbb{R}^m$ denotes the linear space of all $m$ tuples of real numbers.)

Let the input be $u = \{u(t)_i\}_{i \geq 0}$ be quasi-stationary with $u(t) = \pm \alpha$, $t \geq 0$, where $\alpha > 0$, and such that $S_\alpha$ has a bounded inverse in the sense required by Theorem 2.1.

Then, $R_{uv}(0)$ exists and $R_{uv} \in \ell_1(Z)$ and so Theorem 2.1 can be applied to give that the optimal LTI model of the nonlinear system $G$ is the stable LTI system
\[
\begin{align*}
x'(t+1) &= Ax'(t) + B(\alpha)u(t) \\
y'(t) &= Cx'(t).
\end{align*}
\]
(Here, $u$ denotes an arbitrary input, not to be confused with the particular quasi-stationary input used in the model.)

An example of an input that satisfies the above conditions is the following sequence (starting from $u(0), u(1)$, and so on), see [14]
\[
\begin{align*}
&\alpha, -\alpha, \\
&\{\alpha, \alpha, -\alpha, -\alpha, \alpha, -\alpha, -\alpha\} \text{ repeated twice} \\
&\{\alpha, \alpha, \alpha, \alpha, -\alpha, -\alpha; \text{etc.}\} \text{ repeated four times}
\end{align*}
\]
This input has the autocovariance sequence $R_{u\alpha}(0) = \alpha^2, R_{u\alpha}(k) = 0$ for $k \neq 0$. Hence $S^{-1}_\alpha = \text{diag}(\alpha, \alpha^2, \alpha^2, \ldots)$ is indeed a bounded operator as required. (It is quite amazing that this very regular sequence has the same autocovariance sequence as white noise with variance $\alpha^2$.)

Note that the optimal LTI model $y^* = F^*u$ is causal. The residual $e = y - F^*u$ is here quasi-stationary and in fact $R_e = 0$. Thus, if a two-valued input is used in the identification of the aforementioned nonlinear system, then a linear model suffices to model the input–output data exactly. Two-valued inputs are a common practice in system identification when linear models are used. We see that this could lead to the mistaken conclusion that the identified model is very good, when in fact it could result in very bad linear controller designs for the true (unknown) nonlinear system.

Finally, consider the nonlinear system $y = Gu$ (take $A = 0$ and $y = x$ as before) given by
\[
y(t) = b(u(t-1))u(t-1),
\]
where $b(v) = \frac{1}{\sqrt{1 - \alpha^2}} - 1 < v < 1$.

Take $u(t) = \pm \alpha, t \geq 0, 0 < \alpha < 1$ to be quasi-stationary as before. Then
\[
R_{uv}(0) = \frac{\alpha^2}{1 - \alpha^2}.
\]
Furthermore
\[
R_{uv}(k) = \frac{1}{\sqrt{1 - \alpha^2}} R_{uv}(k-1), \quad k \in \mathbb{Z}
\]
and so $R_{uv} \in \ell_1(Z)$ if and only if $R_n \in \ell_1(Z)$.

As a special case let $u$ be the regular two-valued sequence stated earlier in this section to satisfy $R_{uv}(k) = 0$ for $k \neq 0$. Then, $R_{uv}(k) = 0$ for $k \neq 1$ and $R_{uv}(1) = \alpha^2/\sqrt{1 - \alpha^2}$. Hence, the optimal LTI approximation of $G$ is now
\[
y^*(t) = \frac{1}{\sqrt{1 - \alpha^2}} u(t-1).
\]
(Here $y$ denotes again an arbitrary, not necessarily quasi-stationary, input.)

Now, define another sequence $w$ using the aforementioned regular $u$. Put
\[
w(i^2) = 1 - \frac{1}{i^2}, \quad i \geq 1
\]
and $w(t) = u(t)$ for $t \geq 0$ such that $t \neq i^2$ for all $i \geq 1$. Then, it is easy to check that $R_{uw}(k) = R_{uv}(k)$ for any $k \in \mathbb{Z}$. How about the output of the nonlinear system $z = Gu$ corresponding to the input $w$? Does $R_z(0)$ exist? A necessary condition for this is that
\[
\lim_{n \to \infty} \frac{b(w(n))^2 w(n)^2}{n} = 0.
\]
However, for $n = i^2$
\[
\frac{b(w(i^2))^2 w(i^2)^2}{i^2} = \frac{1}{2} - \frac{1}{i^2} \left(1 - \frac{1}{i^2}\right)^2
\]
and so
\[
\lim_{i \to \infty} \frac{b(w(i^2))^2 w(i^2)^2}{i^2} = \frac{1}{2}.
\]
Thus, $R_{zi}(0)$ does not exist. So, two bounded inputs with the same autocovariance sequence applied to the same nonlinear system produce here radically different outputs. This is due to the unbounded nature of the nonlinearity.

IV. CONCLUSION

A least squares LTI approximation problem for nonlinear systems has been solved and an illustrative application of the main result has been provided. In general, quasi-stationarity of the input does not even suffice to guarantee the existence of the root mean square value $\sqrt{R_{uv}(0)}$ of the output. It is a very interesting problem to characterize additional properties of the input such that the output of the nonlinear
system will then be such that the main result of this note can be applied to for example Wiener and Hammerstein systems.

REFERENCES
