A Necessary and Sufficient Stability Criterion for Linear Time-Varying Systems

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Abstract

In this paper a necessary and sufficient exponential stability criterion is presented for linear time-varying (LTV) systems based on a recently developed parallel D-spectrum concept. This new result is a corrected version of some previously published results, where an important condition has been overlooked. Extensions of the new result to the assessment of exponential stability using a series D-spectrum, and the assessment of asymptotic stability using a relative-mean concept for the PD- and SD-eigenvalues are also discussed.

1. Introduction

This paper presents a new necessary and sufficient exponential stability criterion for the class of nth-order scalar linear time-varying (LTV) dynamical systems of the form:

\[ y^{(n)} + \alpha_n(t)y^{(n-1)} + \cdots + \alpha_2(t)y + \alpha_1(t)y = 0 \]  
\[ y^{(k)}(t_0) = y_k, \quad k = 0, 1, \ldots, n-1, \]  

where \( \alpha_i(t) \) are scalar functions, \( y_0, y_1, \ldots, y_{n-1} \) are initial conditions, and \( t_0 \) is a fixed time. The LTV system (1.1) can be conveniently represented as

\[ D_n = \delta^n + \alpha_n(t)\delta^{n-1} + \cdots + \alpha_2(t)\delta + \alpha_1(t) \]  

where \( \delta = \frac{d}{dt} \) is the derivative operator. The study of LTV systems (1.1) is of fundamental importance in the theory of control, communication and dynamic systems. This is the case not only because many dynamical systems can be adequately modeled by linear differential equations (1.1); but also because most of nonlinear dynamical systems can be effectively dealt with by global or local linearization.

It is well-known that for the subclass of linear time-invariant (LTI) systems (1.1) where \( \alpha_n(t) = \alpha_n \) is constant, exponential stability is equivalent to the confinement of all eigenvalues to the left-half-plane (LHP) of \( C \), referred to as the LHP stability criterion. However, as is also well-known, this LHP stability criterion does not carry over, in general, to the time-varying case using the conventional eigenvalue concept.

Recently, a unified spectral (eigenvalue) theory has been developed [1]-[4] for (time-varying) linear dynamic systems (1.1), based on a classical result of Floquet (1879) on the factorization of SPD estimates:

\[ D_n = (\delta - \lambda_n(t)) \cdots (\delta - \lambda_2(t))(\delta - \lambda_1(t)) \]  

In that unified spectral theory a collection \( \{\lambda_k(t)\}_{k=1}^n \) satisfying (1.3) is called a series D-spectrum (SD-spectrum) for \( D_n \), and an \( n \)-parameter family \( \{\rho_k(t) = \lambda_{1,k}(t)\}_{k=1}^n \) is called a parallel D-spectrum (PD-spectrum) for \( D_n \), where \( \lambda_{1,k}(t) \) are particular solutions for \( \lambda_1(t) \) satisfying some nonlinear independence constraints. The scalar functions \( \lambda_k(t) \) and \( \rho_k(t) \) have been called poles and right-poles by Kamen [5], ED- and PED-eigenvalues by Zhu [1], and SD- and PD-eigenvalues by Zhu and Johnson [4].

In this paper a necessary and sufficient stability criterion is presented for the LTV system (1.1) based on a PD-spectrum. This new result is a corrected version of some previously published results [5], [3], where an important condition has been overlooked. Extensions of the main results in this paper to the assessment of exponential stability using an SD-spectrum, and the assessment of asymptotic stability of LTV systems using a relative-mean concept for the PD- and SD-eigenvalues will also be discussed, and will appear in a forthcoming paper.

These new stability criteria have found successful applications in the design of trajectory tracking controllers for time-varying nonlinear dynamic systems and time-varying bandwidth filters [27]. A novelty of these time-varying controllers and filters is that time-variance of system parameters are not treated as nuisances, but rather purposely utilized to achieve performances beyond the reach of conventional LTI systems.
2. Preliminaries

Some technical preliminaries are presented in this section to facilitate development of the main results.

Let \( I \subseteq \mathbb{R} \) be a real interval and let \( G = G(\mathbb{R}) \) (or \( G = G(C) \)) be the differential ring (D-ring) of regulated \( C^\infty \) functions \( f: I \rightarrow \mathbb{R} \) (or \( f: I \rightarrow C \), resp.) with \( \delta = d/dt \) the derivative operator defined on \( K \). Then the SPDO \( D_\alpha \) defined in (1.2) with \( \alpha_0 \in K \) is an operator over the D-ring \( K \). Using the differential operator factorization (1.3), the basic terminology for the unified equation concept can be summarized as follows.

**Definition 2.1.**

(a) Let \( D_\alpha \) be an SPDO operator with \( \alpha_k \in K \), \( k = 1, 2, \ldots, n \). Then, the scalar functions \( \lambda_k \in K \), \( k = 1, 2, \ldots, n \), given by the factorization (1.3) are called Series D-eigenvalues (SD-eigenvalues) of \( D_\alpha \). Moreover, \( \rho(t) = \lambda_1(t) \) is called a Parallel D-eigenvalue (PD-eigenvalue) of \( D_\alpha \).

(b) A multi-set \( \Gamma_\alpha = \{ \lambda_k(t) \}_{k=1}^n \) is called a Series D-spectrum (SD-spectrum) for \( D_\alpha \). (c) A set \( \{ \rho_k \} \subseteq \mathbb{R} \) is called a Parallel D-spectrum (PD-spectrum) for \( D_\alpha \) if \( \rho_k \) are PD-eigenvalues for \( D_\alpha \) and \( \{ \lambda_k = \exp(\int \rho_k(t) \, dt) \}_{k=1}^n \) constitutes a fundamental set of solutions to \( D_\alpha \).

(d) Let \( A_\alpha(t) \) be the companion matrix associated with \( D_\alpha \).

\[
A_\alpha(t) = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 \\
0 & \cdots & 0 & 0 & 1 \\
-\alpha_1(t) & -\alpha_2(t) & \cdots & -\alpha_n(t)
\end{bmatrix}
\]

The matrix \( \Gamma(t) = \begin{bmatrix} \lambda_1(t) & 1 & 0 & \cdots & 0 \\
0 & \lambda_2(t) & 1 & \cdots & 0 \\
0 & \cdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \lambda_{n-1}(t) & 1 \\
0 & \cdots & \cdots & \cdots & \lambda_n(t)\end{bmatrix} \) is called a Series Spectral canonical form (SS canonical form) for \( D_\alpha \). The diagonal matrix \( T(t) = \text{diag}[\rho_1(t), \rho_2(t), \ldots, \rho_n(t)] \) is called a Parallel Spectral canonical form (PS canonical form) for \( D_\alpha \).

(e) Let \( \{ y_i(t) \}_{i=1}^{n+1} \) be any fundamental set of solutions to \( D_\alpha \).

\[
W = \begin{bmatrix}
y_1 & y_2 & \cdots & y_n \\
y_1' & y_2' & \cdots & y_n' \\
\vdots & \vdots & \ddots & \vdots \\
y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)}
\end{bmatrix}
\]

be the Wronskian matrix associated with \( \{ y_i \}_{i=1}^{n+1} \). Denote by \( D \) the diagonal matrix

\[
D = \text{diag}[\rho_1, \rho_2, \ldots, \rho_n].
\]

Then

\[
WD^{-1} = V(\rho_1, \rho_2, \ldots, \rho_n)
\]

\[
= \begin{bmatrix}
1 & 1 & \cdots & 1 \\
D_{\rho_1} \{ 1 \} & \cdots & D_{\rho_n} \{ 1 \} \\
D_{\rho_1} \{ 1 \} & \cdots & D_{\rho_n} \{ 1 \} \\
\vdots & \ddots & \ddots & \ddots \\
D_{\rho_1} \{ 1 \} & \cdots & D_{\rho_n} \{ 1 \}
\end{bmatrix}
\]

where \( D_{\rho_k} = (\delta + \rho_k) \), \( D_{\rho_k}^* = D_{\rho_k} D_{\rho_k}^{-1} \). The canonical coordinate transformation matrix \( V(t) \) is called the Modal canonical matrix for \( D_\alpha \) associated with the D-spectrum \( \{ \rho_i \}_{i=1}^{n} \). The column vectors \( u_i(t) \) of \( V(t) \) satisfying

\[
A_\alpha(t)u_i(t) - \rho_i(t)u_i(t) = u_i(t)
\]

and the row vectors \( u_i^T(t) \) of \( U = V^{-1}(t) \) satisfying

\[
u_i^T(t)A_\alpha(t) - \rho_i(t)u_i^T(t) = -u_i^T
\]

are called column PD-eigenvectors and row PD-eigenvectors, respectively, of \( D_\alpha \) associated with \( \rho_i(t) \). SD-eigenvectors can be defined similarly [3].

The new necessary and sufficient stability criterion based on a PD-spectrum uses an extended-mean concept as defined below.

**Definition 2.2.**

Let \( \sigma : I \rightarrow \mathbb{R} \) be a locally integrable function on \( I = [t_0, \infty) \). The extended mean of \( \sigma(t) \) over \( I \) is defined by

\[
em(\sigma(t)) = \limsup_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^{t_0 + T} \sigma(t) \, dt
\]

3. Main Results

The main results of this paper are presented in this section, which include the main theorem on stability of LTV systems and two lemmas that facilitate proof of the main theorem. Extensions of the new result to the assessment of exponential stability using a series D-spectrum, and the assessment of asymptotic stability using a relative-mean concept for the PD- and SD-eigenvalues are also discussed in a remark following the proof. The main theorem is first stated as follows.

**Theorem.**

Let \( D_\alpha \) be a well-defined \( n \)-th-order scalar polynomial differential operator

\[
D_\alpha = D^n + \alpha_n(t)D^{n-1} + \cdots + \alpha_1(t)D + \alpha_1(t)
\]
where \( D = \frac{d}{dt} \), with a well-defined PD-spectrum \( \{ \rho_k(t) \}_{k=1}^n \) in \( I = [T_0, \infty) \). Let \( v_k(t) \) and \( u_k(t) \) be a column PD-eigenvector and a row PD-eigenvector associated with \( \rho_k(t) \) respectively. Then the null solution to the LTV system \( D_n \{ y \} = 0 \) is uniformly asymptotically stable for all \( t_0 \geq T_0 \) if and only if

(i) there exists \( b > 0 \) such that

\[
\lim_{t \to \infty} \rho_k(t) = 0, \quad k = 1, 2, \ldots, n
\]

and moreover,

(ii) there exist \( h_k > 0 \) and \( 0 < d_k < c_k \) such that

\[
\| v_k(t) u_k^*(t_0) \| < h_k e^{d_k(t-t_0)}
\]

for all \( t \geq t_0 \geq T_0 \).

Remarks.
1. Condition (ii) was overlooked in \([11]\) and \([2]\).
2. Condition (ii) is automatically satisfied if all PD-eigenvalues are of polynomial order or slower; that is, an integer \( m > 0 \) exists such that

\[
| \text{Re} \rho_k(t) | < c_k < 0
\]

3. If \( \lim_{t \to \infty} \rho_k(t) = 0 \) for some \( 1 \leq k \leq n \), then the null solution to \( D_n \{ y \} = 0 \) is unstable. However, if \( \lim_{t \to \infty} \rho_k(t) = 0 \) for some \( 1 \leq k \leq n \), the null solution may be either stable, asymptotically stable, or unstable.

Proof of the Theorem relies on the following three lemmas. The first lemma is a well known result (cf. \([8, p.84]\)), thus its proof is omitted.

Lemma 1.
Let \( D_n \) be a well-defined \( n \)-th order SPDO in \( I = [T_0, \infty) \) with a Wronskian matrix \( W(t) \). Then the null solution to \( D_n \{ y \} = 0 \) is uniformly asymptotically stable in \( I \) if and only if it is exponentially asymptotically stable in \( I \); that is, there are \( b = b(T_0) > 0, a = a(T_0) > 0 \) such that

\[
\| W(t) W^{-1}(t_0) \| \leq be^{-\alpha(t-t_0)}
\]

for all \( t \geq t_0 \geq T_0 \).

The following Lemma 2 constitutes a proof of the main theorem for a 1-st order LTV system with a complex-valued coefficient, since the associated PD-eigenvector is a constant.

Lemma 2.
Let \( \rho(t) \) be a locally integrable, complex-valued function defined in \( I = [T_0, \infty) \). Then

\[
\lim_{t \to \infty} \sigma(t) = c, \quad -\infty \leq c < \infty
\]

if and only if, there exist \( b > 0 \) and \( \alpha > c \) such that

\[
\left| e^{\int_{t_0}^{t_0+T} \sigma(r) dr} \right| < be^{a(t-t_0)} , \quad t \geq t_0 \geq T_0
\]

Proof.
Suppose that inequality (3.2) holds. We have

\[
\left| e^{\int_{t_0}^{t_0+T} \sigma(r) dr} \right| = e^{\int_{t_0}^{t_0+T} \sigma(r) dr} < be^{a(t-t_0)} , \quad t \geq t_0 \geq T_0
\]

This implies that

\[
\int_{t_0}^{t_0+T} \sigma(r) dr < a(t - t_0) + \log b , \quad t \geq t_0 \geq T_0
\]

In particular,

\[
\lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \sigma(r) dr = c < a < 0.
\]

Conversely, suppose (3.1) holds. Then for any \( \epsilon > 0 \) there exists a \( T_1 > 0 \), such that

\[
\int_{t_0}^{t_0+T_1} \sigma(r) dr < aT , \quad T \geq T_1 , \quad t \geq t_0 \geq T_0
\]

where \( a = c + \epsilon \). Let

\[
b = \max_{0 < s \leq \epsilon} | e^{\int_{t_0}^{t_0+T} \sigma(r) dr} |
\]

It then follows that

\[
\left| e^{\int_{t_0}^{t_0+T} \sigma(r) dr} \right| = e^{\int_{t_0}^{t_0+T} \sigma(r) dr} < be^{a(t-t_0)} , \quad t \geq t_0 \geq T_0
\]

Remark.
For time-varying linear systems, an exponentially asymptotically stable dynamic mode \( e^{\rho(t)u} \) may have a satisfactory exponential bound \(-a < 0\), but the scaling factor \( b \) can be impractically large. In such a case, short-time stability concept may be more suitable.

Example.
Let \( T_0 = 0 \), and \( \rho(t) \) be given by

\[
\rho(t) = \begin{cases} 10 & 0 \leq t \leq 1 \\ -1 & 1 < t \end{cases}
\]

Then
Thus the constants \( a, b \) can be chosen as 
\[
a = -0.9, \quad b = e^{10}.
\]

Lemma 3 below allows the preliminary result of Lemma 2 to be extended to the general \( n \)th-order case.

**Lemma 3.**

Let \( D_0 \) be a well-defined \( n \)th-order SPDO with a well-defined PD-spectrum \( \{ \rho_k \}_{k=1}^n \). Let \( v_k(t) \) and \( u_k^T(t) \) be a column PD-eigenvector and a row PD-eigenvector associated with \( \rho_k(t) \), respectively. Define

\[
V(t) = [v_1(t) \mid v_2(t) \mid \cdots \mid v_n(t)]
\]

and

\[
U(t) = [u_1(t) \mid u_2(t) \mid \cdots \mid u_n(t)]^T.
\]

Then, for any given Wronskian matrix \( W(t) \) for \( D_0 \), there exist constants \( r_k, b_k \geq 0 \) and

\[
0 < d_k < c_k < c_k
\]

such that

\[
\|W(t)W^{-1}(t_0)\| = \|V(t)Y(t, t_0)U(t_0)\|
\]

where \( a = \min |d_k - a_k| \), and \( b = \sum_{k=1}^n b_k r_k |b_k| \). It then follows from Lemma 1 that the null solution to \( D_0 \{ y \} = 0 \) is uniformly asymptotically stable in \( I \).

Conversely, suppose that the null solution to \( D_0 \{ y \} = 0 \) is uniformly asymptotically stable in \( I = [t_0, \infty) \). Let \( \{ \rho_k(t) \}_{k=1}^n \) be a well-defined PD-spectrum for \( D_0 \), and let \( W(t) = W(y_1(t), y_2(t), \ldots, y_n(t)) \) be the associated Wronskian matrix, where

\[
y_k(t, t_0) = e^{{\int_{t_0}^t} \rho(r) dr} \neq 0, \quad t \geq t_0 \geq T_0.
\]

**Proof.**

Let \( W(t) \) be a Wronskian matrix for \( D_0 \) and let

\[
Y(t, t_1) = \text{diag}[e^{C_1\rho_1(t_1)dr}, e^{C_2\rho_2(t_1)dr}, \ldots, e^{C_n\rho_n(t_1)dr}]
\]

with an arbitrarily chosen \( t_1 \). Then

\[
\tilde{V}(t) = W(t)Y^{-1}(t)
\]

is a model canonical matrix for \( D_0 \) whose \( k \)th column vector \( \tilde{v}_k(t) \) is a column PD-eigenvector associated with \( \rho_k(t) \), and the \( k \)th row vectors \( \tilde{u}_k^T(t) \) of \( \tilde{U}(t) = \tilde{V}^{-1}(t) \) is a row PD-eigenvector associated with \( \rho_k(t) \). Since PD-eigenvectors span a one dimensional space, the given column PD-eigenvector can be written as \( v_k(t) = \rho_k \tilde{v}_k(t) \), and the given row PD-eigenvector can be written as \( u_k^T = q_k \tilde{u}_k^T(t) \), for some constants \( \rho_k, q_k \). The proof is complete with \( r_k = (\rho_k q_k)^{-1} \).

The main theorem is now readily proved as follows.

**Proof of the Extended-Mean Stability Criterion.**

Suppose that conditions (i) and (ii) hold. Let \( V(t) \), \( Y(t) \) and \( Y(t, t_0) \) be defined as in Lemma 3. By Lemmas 2 and 3, there exist constants \( r_k, b_k > 0 \) and

\[
0 < d_k < a_k < c_k
\]

such that

\[
\|W(t)W^{-1}(t_0)\| = \|V(t)Y(t, t_0)U(t_0)\|
\]

where \( a = \min |d_k - a_k| \), and \( b = \sum_{k=1}^n b_k r_k |b_k| \). It then follows from Lemma 1 that the null solution to \( D_0 \{ y \} = 0 \) is uniformly asymptotically stable in \( I \).

Conversely, suppose that the null solution to \( D_0 \{ y \} = 0 \) is uniformly asymptotically stable in \( I = [t_0, \infty) \). Let \( \{ \rho_k(t) \}_{k=1}^n \) be a well-defined PD-spectrum for \( D_0 \), and let \( W(t) = W(y_1(t), y_2(t), \ldots, y_n(t)) \) be the associated Wronskian matrix, where

\[
y_k(t, t_0) = e^{{\int_{t_0}^t} \rho(r) dr} \neq 0, \quad t \geq t_0 \geq T_0.
\]

Now let \( V(t) = V(\rho_1(t), \rho_2(t), \ldots, \rho_n(t)) \) be the associated modal canonical matrix and let \( U(t) = V^{-1}(t) \). Then the column vectors \( v_k(t) \) of \( V(t) \) and the row vectors \( u_k^T(t) \) of \( U(t) \) are linearly independent column and row PD-eigenvectors associated with \( \rho_k(t) \), respectively. By Lemma 1, we have

\[
\|W(t)W^{-1}(t_0)\| = \left\| \sum_{k=1}^n v_k(t_0) e^{C_k \rho_k(t_1)dr} u_k^T(t_0) \right\|
\]

for some \( a > 0 \) and \( b > 0 \). Since \( \{ y_k(t, t_0) \}_{k=1}^n \), \( \{ v_k(t) \}_{k=1}^n \) and \( \{ u_k^T(t) \}_{k=1}^n \) are all linearly independent sets, it follows that for each \( k = 1, 2, \ldots, n \),

\[
|y_k(t, t_0)| = e^{C_k \rho_k(t_1)dr} \leq e^{-a(t-t_0)}, \quad t \geq t_0 \geq T_0
\]

for some \( \tau > 0 \) and \( a_k > a > 0 \), and

\[
\|v_k(t)u_k^T(t_0)\|e^{C_k \rho_k(t_1)dr} \leq e^{-a(t-t_0)}, \quad t \geq t_0 \geq T_0
\]

By Lemma 2, inequality (3.3) implies that
for some $0 < c_0 < a_0$, this proves Condition (i). To derive Condition (ii), rewrite inequality (3.4) as

$$\|v_k(t)u_k^T(t_0)\| \leq be^{-\int_{t_0}^{T_0} (a + \Re \rho_k(t)) \, dr}$$

Note that

$$\lim_{t \to T_0} (a + \Re \rho_k(t)) = a + c_0 > 0$$

By Lemma 2, there exist $h_k > 0$ and $d_k > 0$ such that

$$c_0 > d_k > c_k - a > 0$$

and

$$\|v_k(t)u_k^T(t_0)\| \leq h_k e^{d_k(t-t_0)}, \quad t \geq t_0 \geq T_0$$

This completes the proof.

Remark.

The results presented in this section are readily extended to the assessment of exponential stability using the extended-mean of an SD-spectrum and the SD-eigenvectors as defined in [3]. If the relative-mean as defined in [3] is used in place of the extended-mean, then asymptotic stability of an LTV system (1.1) can be determined in the same fashion.

4. Summary and Conclusions

In this paper a necessary and sufficient stability criterion has been established for linear time-varying (LTV) systems based on a recently developed parallel D-spectrum concept. This new result is a corrected version of some previously published results [10], [2], where an important condition has been overlooked. Extensions of the main results in this paper to the assessment of exponential stability using the extended-mean of SD-eigenvalues, and the assessment of asymptotic stability of LTV systems using a relative-mean concept for the PD- and SD-eigenvalues have also been discussed, and will appear in a forthcoming paper.

These new stability criteria have found applications in the design of trajectory tracking controllers for time-varying nonlinear dynamic systems and time-varying bandwidth filters [27]. The novelty of these time-varying controllers and filters is that time-variance of system parameters are not treated as nuisances, but rather purposely utilized to achieve performances beyond the reach of conventional LTI systems.

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