4.1 Considerations on time variant linear systems

4.1.1 Properties of the Difference Polynomial Operator

- **Definition 1**: The *time-shift* operator q^{-i} maps ℓ into ℓ and is described by

$$\begin{aligned} q^{-i} : \ell \rightarrow \ell \\ q^{-i}\{x(n)\} &= x(n-i) \end{aligned} \tag{22}$$

- **Property 2**: The time-shift operator is a linear operator, then

$$q^{-i}\{c\,x(n) + d\,y(n)\} = c\,x(n-i) + d\,y(n-i) \tag{23}$$

- **Property 3**: A more general result for the time-shift operator results from its own definition and can be written as

$$q^{-i}\{f(x(n), y(n))\} = f(x(n-i), y(n-i)) \tag{24}$$

  where $f(.,.)$ is any given function defined in the discrete-time domain.

- **Property 4**: The linear combination of time-shift operators is performed as follows

$$(c\,q^{-i} + d\,q^{-j})\{x(n)\} = c\,x(n-i) + d\,x(n-j) \tag{25}$$

- **Property 5**: The concatenation of time-shift operators is performed as follows

$$q^{-j}\{q^{-i}\{x(n)\}\} = q^{-j}\{x(n-i)\} = x(n-i-j) = q^{-(i+j)}\{x(n)\} \tag{26}$$

- **Property 6**: The division of time shift operators follows the rule

$$\left(\frac{q^{-i}}{q^{-i}}\right)\{x(n)\} = q^{(-i-(-j))}\{x(n)\} = x(n-i+j) \tag{27}$$

An important extension, the *difference polynomial operator* (DPO)

- Definition 7: The extrapolation of equation (25) for several terms results in

$$\begin{aligned} \tilde{C}\{x(n)\} &= (c_{n_c} + ... + c_1\, q^{n_c-1} + c_0\, q^{n_c})\{x(n)\} \\ &= c_{n_c}x(n) + ... + c_1 x(n+n_c-1) + c_0 x(n+n_c) \qquad (28) \end{aligned}$$

- With adaptive filters, is common the causal form of the DPO

$$C(q) = q^{-n_c}\tilde{C}(q) \;=\; c_0 + c_1 q^{-1} + ... + cn_c q^{-n_c} \qquad (29)$$

- Then the DPO in the frequency domain $Z\{q^{-i}\{x(n)\}\} = z^{-i}X(z)$, then
- **Property 8**: Extending ideas to transfer function

$$y(n) = H(q)\{x(n)\} \leftrightarrow Y(z) = H(z)X(z) \qquad (30)$$

- **Property 10**: a DPO (one with at least one nonzero coefficient) represents a bijective operator in the subspace of one-sided sequences $x(n)$, such that $x(n) = 0$, $\forall n < 0$.
- **Property 11**: The inverse DPO exists and it is defined by

$$\left(\frac{1}{C(q)}\right)\{x(n)\} = C^{-1}(q)\{x(n)\} \qquad (31)$$

  in such a way that $C^{-1}(q)\{C(q)\{x(n)\}\} = (C^{-1}(q)C(q))\{x(n)\}$.

- **Property 12**: The concatenation of direct and inverse DPOs is a commutative operation, i.e.,

$$\begin{aligned} \left(\frac{1}{C(q)}\right)\left\{\frac{1}{D(q)}\{x(n)\}\right\} &= \left(\frac{1}{D(q)}\right)\left\{\frac{1}{C(q)}\right\} \qquad (32) \\ C(q)\{D(q)\{x(n)\}\} &= D(q)\{C(q))\{x(n)\}\} \qquad (33) \\ \left(\frac{C(q)}{D(q)}\right)\{x(n)\} &= \left(\frac{1}{D(q)}\right)\{C(q)\{x(n)\}\} \\ &= C(q)\left\{\frac{1}{D(q)}\{x(n)\}\right\} \qquad (34) \end{aligned}$$

4.1.2 The Time-varying Difference Polynomial Operator

- **Definition 13**: The TVDPO is defined as

$$\begin{aligned} C(q,n)\{x(n)\} &= (c_0(n) + c_1(n)q^{-1} + ... + c_{n_c}(n)q^{-n_c})\{x(n)\} \\ &= c_0(n)x(n) + c_1(n)x(n-1) + ... + c_{n_c}(n)x(n-n_c) \end{aligned} \quad (35)$$

- **Property 14**: The concatenation of a TVDPO with either a DPO or a TVDPO is not a commutative operation.

- **Example**: Consider the two first-order TVDPOs $C(q,n) = 1 + c_1(n)q^{-1}$ and $D(q,n) = 1 + d_1(n)q^{-1}$ with $C(q,n) \neq D(q,n)$. Defining

$$e_1(n) = C(q,n)\{D(q,n)\{x(n)\}\} \quad (36)$$
$$e_2(n) = D(q,n)\{C(q,n)\{x(n)\}\} \quad (37)$$

it is easy to verify that

$$e_1(n) - e_2(n) = [c_1(n)d_1(n-1) - c_1(n-1)d_1(n)]x(n-2) \quad (38)$$

is generally different from zero, implying that $e_1(n)$ and $e_2(n)$ are two distinct sequences.

- **Property 15**:

$$\begin{aligned} (C(q,n))^2\{x(n)\} &= C(q,n)\{C(q,n)\{x(n)\}\} \\ &= C^2(q,n)\{x(n)\} \end{aligned} \quad (39)$$

$$\begin{aligned} \left(\frac{C(q,n)}{D(q,n)}\right)\{x(n)\} &= \left(\frac{1}{D(q,n)}\right)\{C(q,n)\{x(n)\}\} \\ &\neq C(q,n)\left\{\frac{1}{D(q,n)}\{x(n)\}\right\} \end{aligned} \quad (40)$$

- The difference between both sides of the inequality approximate zero if we assume the coefficients of the TVDPO essentially constant, i.e., the **small step approximation**.

4.1.3 Stability of time varying recursive filters

- Consider the state space description of a time-varying recursive filter

$$\begin{bmatrix} \boldsymbol{x}(n+1) \\ \hat{y}(n) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}(n) & \boldsymbol{b}(n) \\ \boldsymbol{c}(n) & d(n) \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(n) \\ u(n) \end{bmatrix}$$

- To study the small step approximation, consider also a time invariant system with the property that its fixed parameters agree with those of the previous equation at time n, i.e.,

$$\begin{bmatrix} \boldsymbol{A}(k) & \boldsymbol{b}(k) \\ \boldsymbol{c}(k) & d(k) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}(n) & \boldsymbol{b}(n) \\ \boldsymbol{c}(n) & d(n) \end{bmatrix}$$

  for all $k \leq n$.

- If the parameters vary slowly, i.e.,

$$\left\| \begin{bmatrix} \boldsymbol{A}(k) & \boldsymbol{b}(k) \\ \boldsymbol{c}(k) & d(k) \end{bmatrix} - \begin{bmatrix} \boldsymbol{A}(n) & \boldsymbol{b}(n) \\ \boldsymbol{c}(n) & d(n) \end{bmatrix} \right\| \leq \epsilon$$

  for all $k \leq n$ and with ϵ small. An when $\epsilon \rightarrow 0$ the two systems must coincide (in the limit).

- Consider now being approximating a rational system with instantaneous error given, with this approximation, by

$$\begin{aligned} e(n) &= \sum_{k=0}^{\infty}(h_k - \hat{h}_k)u(n-k) \\ &= (h_0 - d)u(n) + \sum_{k=1}^{\infty}(h_k - \boldsymbol{c}\boldsymbol{A}^{k-1}\boldsymbol{b})u(n-k) \end{aligned}$$

where $\begin{bmatrix} \boldsymbol{A}(k) & \boldsymbol{b}(k) \\ \boldsymbol{c}(k) & d(k) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{c} & d \end{bmatrix}$ for all k was used.

- Since the approximating system has time varying coefficients

$$e(n) = (h_0 - d(n))u(n) + \sum_{k=1}^{\infty}(h_k - \boldsymbol{c}(n)\boldsymbol{\Phi}(n, n-k+1)\boldsymbol{b}(n-k))u(n-k)$$

where $\boldsymbol{\Phi}(n, n-k+1) = \begin{cases} \boldsymbol{I} & k = 1; \\ \boldsymbol{A}(n-1)\boldsymbol{A}(n-1)...\boldsymbol{A}(n-k+1) & k > 1. \end{cases}$.

- Clearly both errors are similar if the time-varying parameter change is sufficiently slow.

- If $a_k(n) = 0$, an FIR filter approximation, then the error is

$$e(n) = \sum_{k=0}^{N}(h_0 - b_k(n))u(n-k) + \sum_{k=N+1}^{\infty} h_k u(n-k)$$

- To quantify the similarity of both systems we verify BIBO stability.

- Two remarks:
  - Stability of time-varying IIR filters is a generic necessary condition for a possible parameter updating algorithm to converge.
  - Stability properties of time-varying IIR filters can vary with the specific realization chosen.

- The concept of **exponential stability** and Liapunov methods are helpful to verify BIBO stability.
- If
  - The elements of the state space description remain bounded for all time n;
  - $\boldsymbol{x}(n+1) = \boldsymbol{A}(n)\boldsymbol{x}(n)$ (the homogeneous part) remains exponentially stable.

  the time-varying system is BIBO stable.
- By the second condition, for any bounded initial condition $\|\boldsymbol{x}(n)\| < \infty$, then $\|\boldsymbol{x}(m)\| \leq \beta\,\alpha^{m-n}\,\|\boldsymbol{x}(n)\|$, for all $m \geq n$, where $\beta > 0$ and $0 \leq \alpha < 1$.
- Then using $\boldsymbol{\Phi}(m,n)$ we obtain

$$\boldsymbol{x}(m) \;=\; \boldsymbol{\Phi}(m,n)\boldsymbol{x}(n) + \sum_{k=n}^{m-1} \boldsymbol{\Phi}(m-1,k)\boldsymbol{b}(k)u(k)$$

  for all $m > n$, from where it is not hard to see that $\boldsymbol{x}(m)$ remains bounded, and then $\hat{y}(n)$, which implies BIBO stability.

- A constructive way to shown exponential stability: **Liapunov method**, can lead to BIBO stability without appeal to the small step approximation or slow parameter variation approximation.
- To quantify the slow parameter approximation we consider

$$\|\boldsymbol{A}(n+1) - \boldsymbol{A}(n)\| \leq \epsilon$$

or $\boldsymbol{x}(n+N) = \boldsymbol{A}(n+N-1)...\boldsymbol{A}(n+1)\boldsymbol{A}(n)\boldsymbol{x}(n)$ is approximated by $\overline{\boldsymbol{x}}(n+N) = \boldsymbol{A}^N\overline{\boldsymbol{x}}(n)$.

- But this is stable if $\boldsymbol{A}(n)$ has all its eigenvalues inside the unit circle. In particular, this is true if exist $\boldsymbol{P} > 0$ for $\boldsymbol{P}_2 = \boldsymbol{P} - (\boldsymbol{A}^N)^T\boldsymbol{P}\boldsymbol{A}^N > 0$.
- Then for the fixed case, this leads to

$$\begin{aligned} \overline{\boldsymbol{x}}^T(n)\boldsymbol{P}\overline{\boldsymbol{x}}(n) - \overline{\boldsymbol{x}}^T(n+N)\boldsymbol{P}\overline{\boldsymbol{x}}(n+N) &= \overline{\boldsymbol{x}}^T(n)\boldsymbol{P}_2\overline{\boldsymbol{x}}(n) \\ \|\overline{\boldsymbol{x}}(n)\|^2_{\boldsymbol{P}} - \|\overline{\boldsymbol{x}}(n+N)\|^2_{\boldsymbol{P}} &= \|\overline{\boldsymbol{x}}(n)\|^2_{\boldsymbol{P}_2} \geq c_1\|\overline{\boldsymbol{x}}(n)\|^2_{\boldsymbol{P}} \end{aligned}$$

where $c_1\|\boldsymbol{v}\|_{\boldsymbol{P}} \leq \|\boldsymbol{v}\|_{\boldsymbol{P}_2} \leq c_2\|\boldsymbol{v}\|_{\boldsymbol{P}}$ was used. Then

$$\|\overline{\boldsymbol{x}}(n+N)\|^2_{\boldsymbol{P}} \leq (1-c_1)\|\overline{\boldsymbol{x}}(n)\|^2_{\boldsymbol{P}}$$

- The small step approximation leads to

$$\|\overline{\boldsymbol{x}}(n+N) - \boldsymbol{x}(n+N)\|^2_{\boldsymbol{P}} \leq \delta$$

where δ is a constant forced to be small if ϵ is small. This then must lead to

$$\|\boldsymbol{x}(n+N)\|_{\boldsymbol{P}} < \|\boldsymbol{x}(n)\|_{\boldsymbol{P}}$$

- A constructive way to verify the exponential stability of time-varying systems is using the Liapunov equation.
- Consider $\{\boldsymbol{A}(.)\}$ related to our time varying system, that satisfy $\boldsymbol{P} - \boldsymbol{A}^T(n)\boldsymbol{P}\boldsymbol{A}(n) = \boldsymbol{C}(n)\boldsymbol{C}^T(n) > 0$.
- Define $\boldsymbol{y}(n) = \boldsymbol{C}^T(n)\boldsymbol{x}(n)$.
- Following similar ideas than previously, and considering a time window of N time instants

$$\begin{aligned}
\|\boldsymbol{x}(n)\|^2_{\boldsymbol{P}} - \|\boldsymbol{x}(n+1)\|^2_{\boldsymbol{P}} &= \|\boldsymbol{y}(n)\|^2_{\boldsymbol{I}} \\
\|\boldsymbol{x}(n+1)\|^2_{\boldsymbol{P}} - \|\boldsymbol{x}(n+2)\|^2_{\boldsymbol{P}} &= \|\boldsymbol{y}(n+1)\|^2_{\boldsymbol{I}} \\
&\vdots \\
\|\boldsymbol{x}(n+N-1)\|^2_{\boldsymbol{P}} - \|\boldsymbol{x}(n+N)\|^2_{\boldsymbol{P}} &= \|\boldsymbol{y}(n+N-1)\|^2_{\boldsymbol{I}}
\end{aligned}$$

Summing

$$\|\boldsymbol{x}(n)\|^2_{\boldsymbol{P}} - \|\boldsymbol{x}(n+N)\|^2_{\boldsymbol{P}} = \sum_{k=n}^{N+M-1} \|\boldsymbol{y}(k)\|^2_{\boldsymbol{I}}$$

- Note that

$$\begin{aligned}
\begin{bmatrix} \boldsymbol{y}(n) \\ \boldsymbol{y}(n+1) \\ \vdots \\ \boldsymbol{y}(n+N-1) \end{bmatrix} &= \begin{bmatrix} \boldsymbol{C}^T(n) \\ \boldsymbol{C}^T(n+1)\boldsymbol{A}(n) \\ \vdots \\ \boldsymbol{C}^T(n+N-1)\boldsymbol{\Phi}(n+N-1,n) \end{bmatrix} \boldsymbol{x}(n) \\
&= \mathcal{O}(n+N-1,n)\boldsymbol{x}(n)
\end{aligned}$$

- Also note that

  Definition: The pair $(\boldsymbol{A}(.), \boldsymbol{C}^T(.))$ over the time window $n, n+1, ..., m$ is **uniformly observable** if there exists an integer N and positive constants c_1 and c_2 such that

$$0 < c_1\boldsymbol{I} \leq \mathcal{O}^T(n+N-1,n)\mathcal{O}(n+N-1,n) \leq c_2\boldsymbol{I} < \infty$$

- Then, if this property is verified

$$\begin{aligned}\|\boldsymbol{x}(n)\|^2_{\boldsymbol{P}} - \|\boldsymbol{x}(n+N)\|^2_{\boldsymbol{P}} &= \boldsymbol{x}^T(n)\mathcal{O}^T(n+N-1,n)\mathcal{O}(n+N-1,n)\boldsymbol{x}(n)\\ &\geq c_1\|\boldsymbol{x}(n)\|^2_{\boldsymbol{I}} \geq c_1c_3\|\boldsymbol{x}(n)\|^2_{\boldsymbol{P}}\end{aligned}$$

  or

$$\|\boldsymbol{x}(n+N)\|^2_{\boldsymbol{P}} \;\geq\; (1-c_1c_3)\|\boldsymbol{x}(n)\|^2_{\boldsymbol{P}}$$

  Note that $0 \leq (1-c_1c_3) < 1$ since $\|\boldsymbol{x}(n+N)\|^2_{\boldsymbol{P}}$ is positive.

- Increasing n by N steps successively we obtain

$$\|\boldsymbol{x}(n+kN)\|^2_{\boldsymbol{P}} \;\geq\; (1-c_1c_3)^k\|\boldsymbol{x}(n)\|^2_{\boldsymbol{P}}$$

  but the "worst" case decay is to maintain constant $\|\boldsymbol{x}(n)\|^2_{\boldsymbol{P}}$ by N iterations.

- Then, this leads to the following equation

$$\|\boldsymbol{x}(m)\|^2_{\boldsymbol{P}} \;\geq\; \beta^2(1-c_1c_3)^{(m-n)/N}\|\boldsymbol{x}(n)\|^2_{\boldsymbol{P}}$$

  for all $m > n$, for some constant β. This is nothing more than the exponential equation related to stability by recognizing that $\alpha = (1-c_1c_3)^{1/2N}$.

- To summarize

  Theorem: The homogeneous time varying system $\boldsymbol{x}(n+1) = \boldsymbol{A}(n)\boldsymbol{x}(n)$ is exponentially stable provided there exists a symmetric positive definite matrix $\boldsymbol{P}$ fulfilling a Liapunov equation

$$\boldsymbol{P} - \boldsymbol{A}^T(n)\boldsymbol{P}\boldsymbol{A}(n) = \boldsymbol{C}(n)\boldsymbol{C}^T(n)$$

  such that the resultant sequence $\{\boldsymbol{C}(.)\}$ gives $(\boldsymbol{A}(.), \boldsymbol{C}^T(.)\}$ as a uniformly observable pair.

4.2 The Ordinary Difference equation

4.2.1 ODE association to IIR Adaptive Algorithms

The ODE method can be used in adaptive algorithms of the following general form

$$\begin{aligned} \boldsymbol{\theta}(n+1) &= \boldsymbol{\theta}(n) + \alpha \tilde{\boldsymbol{R}}^{-1}(n+1)\boldsymbol{\Psi}(n)e(n) \\ \tilde{\boldsymbol{R}}(n+1) &= \tilde{\boldsymbol{R}}(n) + \alpha[\boldsymbol{\Psi}(n)\boldsymbol{\Psi}^H(n) - \tilde{\boldsymbol{R}}(n)] \end{aligned} \tag{41}$$

where $\boldsymbol{\theta}(n)$ is the parameter vector, $\boldsymbol{\Psi}(n)$ is the regressor, and $e(n)$ is the prediction error.

The average behavior of $\boldsymbol{\theta}(n)$ and $\tilde{\boldsymbol{R}}(n)$ in the previous algorithm can be studied by the solution of the following associated ODE

$$\begin{aligned} \frac{\partial \boldsymbol{\vartheta}(t)}{\partial t} &= \boldsymbol{\varrho}^{-1}(t)E[\boldsymbol{\Psi}(n)e(n)] \\ \frac{\partial \boldsymbol{\varrho}(t)}{\partial t} &= E[\boldsymbol{\Psi}(n)\boldsymbol{\Psi}^T(n)] - \boldsymbol{\varrho}(t) \end{aligned} \tag{42}$$

In order to justify this association, there exist two behaviors that define the convergence analysis of interest:

- constant α (weak convergence or convergence in distribution).
- decrescent α (convergence with probability one).

The conditions for ODE association for both cases are summarized as follows.

- decrescent α.
  - a) $\alpha(n) \rightarrow 0$ for $n \rightarrow \infty$, i.e., the convergence is possible independently of the stochastic environment.
  - b) $\sum_{n=1}^{\infty} \alpha(n) = \infty$, i.e., the estimate is reached through an arbitrary number of iterations.

  A typical example is $\alpha(n) = 1/n$. Note that $t = \sum_{k=1}^{n} \alpha(k)$.

  1. Smoothness
  2. Regularity
  3. Boundness
  4. Liapunov Function.

- α constant.
  In this case $t = n\alpha$, and $t = n\alpha \rightarrow \infty$ for $\alpha \rightarrow 0$

  1. 1 to 3.: Analogous to the previous case.
  2. Stationarity.

Some properties of the analysis by the ODE associated

- $\boldsymbol{\vartheta}(t)$ converge to the stable stationary points of equation (42).
- The ODE trajectories determine an average asymptotic path for the estimate $\boldsymbol{\theta}(n+1)$.
- A drawback of the analysis by the ODE method is that the information related to convergence speed is lost.

4.2.2 Heuristics to ODE approximation

Defining $\boldsymbol{\nabla}(n)$ as the regressor vector and $\boldsymbol{p}(n)$ as the parameter vector, the stochastic gradient version of the updating equation for can be written as follows

$$\boldsymbol{p}(n+1) = \boldsymbol{p}(n) + \mu e(n, \{\boldsymbol{p}(n)\})\boldsymbol{\nabla}(n, \{\boldsymbol{p}(n)\})$$

For N successive time instants (N sufficiently large)

$$\begin{aligned}
\boldsymbol{p}(n+1)-\boldsymbol{p}(n) &= \mu e(n,\{\boldsymbol{p}(n)\})\boldsymbol{\nabla}(n,\{\boldsymbol{p}(n)\}) \\
\boldsymbol{p}(n+2)-\boldsymbol{p}(n+1) &= \mu e(n+1,\{\boldsymbol{p}(n+1)\})\boldsymbol{\nabla}(n+1,\{\boldsymbol{p}(n+1)\}) \\
&\cdots \\
\boldsymbol{p}(n+N)-\boldsymbol{p}(n+N-1) &= \mu e(n+N-1,\{\boldsymbol{p}(n+N-1)\})\boldsymbol{\nabla}(n+N-1,\{\boldsymbol{p}(n+N-1)\})
\end{aligned}$$

or

$$\boldsymbol{p}(n+N) - \boldsymbol{p}(n+N-1) = \mu \sum_{k=n}^{n+N-1} e(k, \{\boldsymbol{p}(k)\})\boldsymbol{\nabla}(k, \{\boldsymbol{p}(k)\})$$

If μ is sufficiently small (i.e., the small step approximation) such that

$$\boldsymbol{p}(n) \cong \boldsymbol{p}(n+1)... \cong \boldsymbol{p}(n+N-1)$$

then

$$\begin{aligned}
e(k, \{\boldsymbol{p}(k)\}) &\cong e(k/\boldsymbol{p}(n)) \\
\boldsymbol{\nabla}(k, \{\boldsymbol{p}(k)\}) &\cong \boldsymbol{\nabla}(k/\boldsymbol{p}(n))
\end{aligned}$$

for $k \geq n$, i.e., each output of these filters is considered an stationary process.

Indeed, if the reference and the input signals are stationary then $e(.)$ and $\boldsymbol{\nabla}(.)$ are stationary if $\boldsymbol{p}(.)$ is in a **well defined domain.**

Also, a natural approximation (ergodic assumption)

$$\sum_{k=n}^{n+N-1} e(k/\boldsymbol{p}(n))\boldsymbol{\nabla}(k/\boldsymbol{p}(n)) \cong NE[e(n/\boldsymbol{p}(n))\boldsymbol{\nabla}(n/\boldsymbol{p}(n))]$$

or

$$\frac{\boldsymbol{p}(n+N)-\boldsymbol{p}(n)}{N} \cong E[e(n/\boldsymbol{p}(n))\boldsymbol{\nabla}(n/\boldsymbol{p}(n))]$$

Introducing now a change of variables, specifically: a continuous time t, such that $t = n$ and $\Delta t = \mu N$, we obtain

$$\frac{\boldsymbol{p}(t+\Delta t)-\boldsymbol{p}(t)}{\Delta t} \cong E[e(n/\boldsymbol{p}(n))\boldsymbol{\nabla}(n/\boldsymbol{p}(n))] \triangleq f(\boldsymbol{p})$$

And finally, for $\mu << 1$ or $\Delta t \to 0$,

$$\frac{\partial \boldsymbol{p}(t)}{\partial t} = f(\boldsymbol{p})$$

4.2.3 Stability analysis

Let $\boldsymbol{p}_*$ a convergence point of the ODE, then

- $\boldsymbol{p}_*$ is an **stationary point**, if $f(\boldsymbol{p}_*) = 0$ such that $\frac{\partial \boldsymbol{p}(t)}{\partial t} = 0$, or

$$\{\boldsymbol{p}_*\} = \{\boldsymbol{p} \ : \ f(\boldsymbol{p}) = 0\}$$

- $\boldsymbol{p}_*$ is an **stable stationary point** (attractor), when

$$\forall \boldsymbol{p}(0), \ such\ that |\boldsymbol{p}_* - \boldsymbol{p}(0)| < \epsilon, \ then\ \boldsymbol{p}(t) \overset{t\to\infty}{\to} \boldsymbol{p}_*$$

- if $\boldsymbol{p}_*$ not satisfies the previous condition is is an **unstable stationary point**.

Liapunov direct method

- Exist $L(\boldsymbol{p}) > 0$ a scalar function, with local minimum in $\boldsymbol{p}_*$, i.e,

$$L(\boldsymbol{p}(t)) > L(\boldsymbol{p}_*) \geq 0$$

$\forall \boldsymbol{p}(t) \neq \boldsymbol{p}_*$ *such that* $|\boldsymbol{p}(t) - \boldsymbol{p}_*| \leq \epsilon$.

- $L(\boldsymbol{p}(t))$ decrescent in all trajectories of $\boldsymbol{p}(t)$, i.e.

$$\frac{\partial L(\boldsymbol{p}(t))}{\partial t} = \frac{\partial L(\boldsymbol{p}(t))}{\partial \boldsymbol{p}(t)}^t \frac{\partial \boldsymbol{p}(t)}{\partial t} = \frac{\partial L(\boldsymbol{p}(t))}{\partial \boldsymbol{p}(t)}^t f(\boldsymbol{p}(t)) \leq 0$$

$\forall \boldsymbol{p}(t) \neq \boldsymbol{p}_*$ *such that* $|\boldsymbol{p}(t) - \boldsymbol{p}_*| \leq \epsilon$.

then $\boldsymbol{p}_*$ is a local stationary point of the ODE.

Liapunov indirect method

Linearization around a stationary point, i.e.

$$f(\boldsymbol{p}_* + \Delta \boldsymbol{p}) = f(\boldsymbol{p}_*) + \left[\frac{\partial f(\boldsymbol{p})}{\partial \boldsymbol{p}}\right]_{\boldsymbol{p}=\boldsymbol{p}_*} \Delta \boldsymbol{p} + O(\Delta \boldsymbol{p})\Delta \boldsymbol{p}$$

then p_* is a local stationary point if $\frac{\partial \boldsymbol{p}(t)}{\partial t} = \left[\frac{\partial f(\boldsymbol{p})}{\partial \boldsymbol{p}}\right]_{\boldsymbol{p}=\boldsymbol{p}_*}$ has all their eigenvalues with negative real part.

4.3 Some linear systems concepts

- Consider the **state variable** equation

$$\begin{bmatrix} \boldsymbol{x}(n+1) \\ \hat{y}(n) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{c} & d \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(n) \\ u(n) \end{bmatrix}$$

then $\hat{H}(z) = d + \boldsymbol{c}(z\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b} = \sum_{k=0}^{\infty} \hat{h}_k z^{-k}$ with $\hat{h}_k = \begin{cases} d, & k=0; \\ \boldsymbol{c}\boldsymbol{A}^{k-1}\boldsymbol{b}, & k=1,2\ldots \end{cases}$.

- The **controllability and observability** grammians fulfill

$$\begin{aligned} \boldsymbol{K} &= \boldsymbol{A}\boldsymbol{K}\boldsymbol{A}^T + \boldsymbol{b}\boldsymbol{b}^T \\ \boldsymbol{W} &= \boldsymbol{A}\boldsymbol{W}\boldsymbol{A}^T + \boldsymbol{c}^T\boldsymbol{c} \end{aligned}$$

where $\boldsymbol{K}$ (respectively $\boldsymbol{W}$) is definite positive if and only if $(\boldsymbol{A}, \boldsymbol{b})$ (resp. $(\boldsymbol{A}, \boldsymbol{c})$) is completely controllable (resp. completely observable), or $\hat{H}(z)$ is **minimal**.

This result is closely related to the "infinite horizon" controllability (resp. observability) matrix

$$\begin{aligned} \mathcal{C} &= [\boldsymbol{b}\ \boldsymbol{A}\boldsymbol{b} \cdots \ \boldsymbol{A}^{N-1}\boldsymbol{b} \cdots \ \boldsymbol{A}^{N+k}\boldsymbol{b} \cdots] \\ \mathcal{O} &= [\boldsymbol{c}\ \boldsymbol{c}\boldsymbol{A} \cdots \ \boldsymbol{c}\boldsymbol{A}^{N-1} \cdots \ \boldsymbol{c}\boldsymbol{A}^{N+k} \cdots]^T \end{aligned}$$

since $\boldsymbol{K} = \mathcal{C}\mathcal{C}^T$ (resp. $\boldsymbol{W} = \mathcal{O}^T\mathcal{O}$).

Also, driving with a unit-variance white noise sequence the state variable filter, it is possible to obtain

$$E\{\boldsymbol{x}(n)\boldsymbol{x}^T(n)\} \;=\; \boldsymbol{A}E\{\boldsymbol{x}(n)\boldsymbol{x}^T(n)\}\boldsymbol{A}^T + \boldsymbol{b}\boldsymbol{b}^T$$

in such a way that $\boldsymbol{K}$ represent the **state covariance matrix**.

- Consider the following double infinite **Hankel form matrix**

$$\boldsymbol{\Gamma}_{\hat{H}} \;=\; \begin{bmatrix} \hat{h}_1 & \hat{h}_2 & \hat{h}_3 & \cdots \\ \hat{h}_2 & \hat{h}_3 & \hat{h}_4 & \cdots \\ \hat{h}_3 & \hat{h}_4 & \hat{h}_5 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

then $\boldsymbol{\Gamma}_{\hat{H}} = \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{cA} \\ \boldsymbol{cA}^2 \\ \vdots \end{bmatrix} [\boldsymbol{b}\ \boldsymbol{Ab}\ \boldsymbol{A}^2\boldsymbol{b}\ \cdots] = \mathcal{OC}$ has rank N if the realization is minimal.

Note also that, their eigenvalues (or singular values) verify

$$\begin{aligned} \sigma_k(\boldsymbol{\Gamma}_{\hat{H}}) &= \sqrt{\lambda_k(\boldsymbol{\Gamma}_{\hat{H}}^T\boldsymbol{\Gamma}_{\hat{H}})} \\ &= \sqrt{\lambda_k(\mathcal{C}^T\mathcal{O}^T\mathcal{OC})} \\ &= \sqrt{\lambda_k(\mathcal{O}^T\mathcal{OCC}^T)} \\ &= \sqrt{\lambda_k(\boldsymbol{KW})} \end{aligned}$$

> Theorem: The Hankel form Γ_H is of finite rank N if and only if $H(z)$ is an N-th order rational function. Then, for $\sigma_1(\Gamma_H) \geq \sigma_2(\Gamma_H) \geq \sigma_3(\Gamma_H) \geq \cdots$, rank $(\Gamma_H) = N$ if and only if $\sigma_N > 0$ and $\sigma_{N+1} = 0$.

then if $\boldsymbol{K} = \boldsymbol{W} = diag(\sigma_1, ..., \sigma_N)$ the realization is **internally balanced**.

- Let $x(n) = \frac{1}{A(z)}u(n)$ an N-order autoregressive process (i.e., with $u(n)$ white noise), then

$$E\left\{\boldsymbol{x}(n)\boldsymbol{x}^T(n)\right\} = \left(\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & 0 & \cdots & 0 \\ \vdots & a_1 & \ddots & \ddots & \vdots \\ a_N & \vdots & \ddots & a_1 & 1 \end{bmatrix}[.]^T - \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ a_N & 0 & 0 & \cdots & 0 \\ \vdots & a_N & \ddots & \ddots & \vdots \\ a_2 & a_3 & \cdots & 0 & 0 \\ a_1 & a_2 & \cdots & a_N & 0 \end{bmatrix}[.]^T\right)^{-1}$$

- Orthonormal realizations:

  Theorem: Let $V(z)$ be an stable all-pass transfer function of McMillan degree N, and let $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{g}, \nu_0)$ be a balanced realization of $V(z)$. Denote $F_k(z) = z(z\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}V^k(z)$. Then $\{\boldsymbol{e}_i^T F_k(z)\}_{i=1,\ldots,N;k=0,\ldots\infty}$ constitutes an orthonormal basis of the space $\mathcal{H}_2$ (stable and causal functions).

  Corollary: for every proper stable transfer function $\hat{H}(z) \in \mathcal{H}_2$ exist unique d and $\boldsymbol{\nu} = \{\nu_k\}_{k=0,1,\ldots} \in l_2$ such that $\hat{H}(z) = \nu_0 + z^{-1}\sum_{k=0}^{\infty} \nu_k F_k(z)$. Then ν_0, ν_k are the **orthogonal expansion coefficients** of $\hat{H}(z)$.

Particular cases

- Obviously, if $V(z) = z^{-1}$, it corresponds to $\nu_k = \boldsymbol{c}\boldsymbol{A}^k\boldsymbol{b}$.
- If $V(z) = \frac{1-az}{z-a}$, with some real-valued a, $|a| < 1$ with $\alpha = 1-a^2$, then $F_k(z) = \alpha z\frac{(1-az)^k}{(z-a)^{k+1}}$, that corresponds to discrete-time **Laguerre functions**.
- An orthonormal extension with real-valued poles

$$F_k(z) = \frac{\alpha_k z}{z - a_{k+1}} \prod_{i=1}^{k-1} \left(\frac{1 - a_i z}{z - a_i}\right) \tag{43}$$

where a_k is the k-th pole and $\alpha_k = 1-a_k^2$ is a normalization constant.

– If $F_k(z)$ are the Szego polynomials $\overline{D}_k(z)$, that related to the direct for realization, $\hat{H}(z) = \frac{B(z)}{A(z)} = \sum_{k=0}^{N} \nu_k \frac{\overline{D}_k(z)}{D_N(z)}$, then the **normalized lattice** realization can be obtained.

– By the state space description

$$\begin{bmatrix} \boldsymbol{x}(n+1) \\ w(n) \end{bmatrix} = \boldsymbol{Q} \begin{bmatrix} \boldsymbol{x}(n) \\ u(n) \end{bmatrix} \qquad \hat{y}(n) = [\nu_0 \; nu_1 \; ... \; \nu_N] \begin{bmatrix} \boldsymbol{x}(n+1) \\ w(n) \end{bmatrix}$$

where $\boldsymbol{Q} = \boldsymbol{Q}_1 \boldsymbol{Q}_2 ... \boldsymbol{Q}_N$ is orthogonal and

$$\boldsymbol{Q}_k = \begin{bmatrix} \boldsymbol{I}_{k-1} & & & \\ & -\sin\theta_k & \cos\theta_k & \\ & \cos\theta_k & \sin\theta_k & \\ & & & \boldsymbol{I}_{N-k} \end{bmatrix}$$

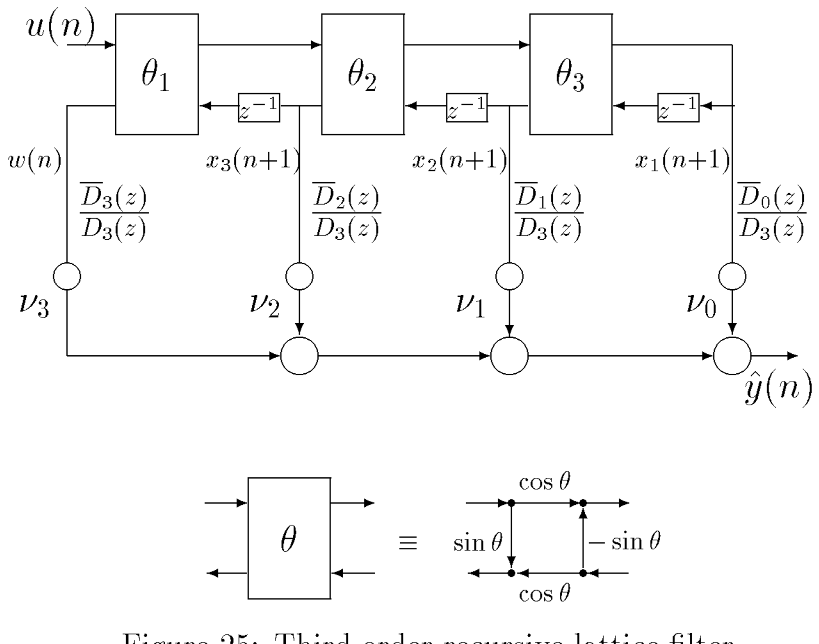

Figure 25: Third order recursive lattice filter.

4.4 Rational approximation theory

4.4.1 Some definitions

We denote $f(z) \in L_2$ to mean

- $\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(e^{jw})|^2 dw < \infty$, that admits a Fourier series $f(e^{jw}) = \sum_{k=-\infty}^{\infty} f_k e^{jkw}$ such that $\sum_{k=-\infty}^{\infty}|f_k|^2 < \infty$.
- Also, if $F(z) = \sum_{k=-\infty}^{\infty} f_k z^{-k}$ and $G(z) = \sum_{k=-\infty}^{\infty} g_k z^{-k}$ for $|z| = 1$, then the inner product becomes

$$< F(z), G(z) > \; = \; \frac{1}{2\pi j}\int_{|z|=1} F(z^{-1})G(z)\frac{dz}{z} = \sum_{k=-\infty}^{\infty} f_k g_k$$

We denote $f(z) \in \mathcal{H}_p$ if

- For $1 \leq p < \infty$, $\|f(e^{jw})\| = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(e^{jw})|^p dw\right)^{1/p} < \infty$.
  For $p \to \infty$, $\|f(e^{jw})\|_\infty = \sup_w |f(e^{jw})| < \infty$.
- The Fourier series expansion is one-sided: $f(e^{jw}) = \sum_{k=0}^{\infty} f_k e^{jkw}$. This implies that $f(e^{jw})$ can be analytically continued to all points outside the unit circle by $f(z) = \sum_{k=0}^{\infty} f_k z^{-k}$, $|z| > 1$.

In particular, "$f(z)$ stable and causal" and "$f(z) \in \mathcal{H}_2$" may be used interchangeably.

Note: If $f(z) \in L_2$ but not fully in $\mathcal{H}_2$ we can write $f(z) = [f(z)]_- + [f(z)]_+$, the sum of the anti-causal and strict causal part. Also, for $f(z), g(z) \in L_2$, $< [f(z)]_-, [g(z)]_+ >= 0$.

4.4.2 Decomposition of $\mathcal{H}_2$

The doubly infinite Hankel form Γ_H, the matrix representation of a rational system, defines two important $\mathcal{H}_2$ subspaces, called **shift-invariant subspaces**, left shift-invariant subspace (lsis) and right shift-invariant subspace (rsis), i.e., given

$$[g_1\, g_2\, g_3\, ...] \;=\; [f_0\, f_1\, f_2\, ...]\begin{bmatrix} h_1 & h_2 & h_3 & \cdots \\ h_2 & h_3 & h_4 & \cdots \\ h_3 & h_4 & h_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

a right shift to $f(z)$ gives

$$[g_2\, g_3\, g_4\, ...] \;=\; [0\, f_0\, f_1\, ...]\begin{bmatrix} h_1 & h_2 & h_3 & \cdots \\ h_2 & h_3 & h_4 & \cdots \\ h_3 & h_4 & h_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

that results in a left shift to $g(z)$.

- Since Γ_H is not necessarily square it can have a range space and a non empty null space. Since Γ_H is symmetric its range and null space are orthogonal.
- If $g(z)$ is in the range space of Γ_H so is its left shifted version. Also if $f(z)$ is in the null space of Γ_H so must its right shifted version.

Revisiting orthogonal realizations based on shift invariant subspaces

Theorem:

- To every *rsis* is associated a unique all-pass function, $V(z)$, which causally divides every element of the *rsis*. Thus each *rsis* may be written as $V(z)\mathcal{H}_2$ to denote the set of functions $V(z)f(z)$ as $f(z)$ varies over $\mathcal{H}_2$;
- Since every *lsis* is the orthogonal complement of a *rsis*, each *lsis* may be written $\mathcal{H}_2 \ominus V(z)\mathcal{H}_2$, to denote the orthogonal complement to $V(z)\mathcal{H}_2$;
- The dimension of the subspace $\mathcal{H}_2 \ominus V(z)\mathcal{H}_2$ is the McMillan degree of $V(z)$.
- If degree $V(z) = N$, then a set of linearly independent basis functions for the subspace $\mathcal{H}_2 \ominus V(z)\mathcal{H}_2$ may be taken as the N functions of the transfer vector $\mathcal{C}(z) = z(z\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}$, where the $N \times N$ matrix $\boldsymbol{A}$ and the $N \times 1$ vector $\boldsymbol{b}$ are such the pair $(\boldsymbol{A}, \boldsymbol{b})$ is completely controllable and the eigenvalues of $\boldsymbol{A}$ coincide with the zeros of $V(z)$.

In particular, suppose $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{g}, \nu_0)$ is an orthogonal realization, then

Lemma: Let $\mathcal{C}(z)$ the controllability transfer vector, and let $V(z) = \frac{\det(\boldsymbol{I}-z\boldsymbol{A})}{\det(z\boldsymbol{I}-\boldsymbol{A})}$, then

- $< z^{-k}V(z), z^{-l}V(z) >= \delta_{kl}$,
- $< \mathcal{C}(z), z^{-k}V(z) >= \mathbf{0}$ for all $k \geq 0$.
- $f(z) \in \mathcal{H}_2$ satisfies $< \mathcal{C}(z), f(z) >= \mathbf{0}$ if and only if $f(z)$ is causally divisible by $V(z)$, i.e.,

$$f(z) \;=\; V(z)g(z) \quad \textit{for some } g(z) \in \mathcal{H}_2$$

4.4.3 Relation with Hankel form

Problem: Given h_k, find the parameters of a rational description (with finite unknown degree).

In terms of the usual transfer function operator, the matrix Hankel form

$$\begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} h_1 & h_2 & h_3 & \cdots \\ h_2 & h_3 & h_4 & \cdots \\ h_3 & h_4 & h_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \end{bmatrix}$$

i.e., $\boldsymbol{g} = \boldsymbol{\Gamma}_H \boldsymbol{f}$, with $g(z)$ strictly casual and $f(z)$ casual, can be rewritten as

$$g(z) = [H(z)f(z^{-1})]_+$$

where $[.]_+$ is the strictly causal projection operator.

Let $\boldsymbol{\Gamma}_H$ of finite rank N, then

Theorem: Let $\mathcal{N}(\boldsymbol{\Gamma}_H) = \{\boldsymbol{f} : \boldsymbol{\Gamma}_H \boldsymbol{f} = \boldsymbol{0}\}$ denote the set of vectors $\boldsymbol{f} = [f_0, f_1, f_2, ...]^T$ lying in the null space of $\boldsymbol{\Gamma}_H$, or equivalently

$$\mathcal{N}(\boldsymbol{\Gamma}_H) = \{f(z) : [H(z)f(z^{-1})]_+ = 0\}$$

Then

- exist a causal all-pass $V(z)$, determined by $H(z)$, such that $f(z) = \sum_{k=0}^{\infty} f_k z^{-k} = V(z)R(z)$ for some $R(z) \in \mathcal{H}_2$.
- Since $H(z) = \frac{B(z)}{A(z)}$ then $V(z) = \frac{z^{-N}A(z^{-1})}{A(z)}$.

Particular interesting cases can be obtained by chosen $R(z)$ as follows

1. $R(z) = A(z)$, then $f(z) = z^{-N}A(z^{-1})$ a finite length sequence (equation error methods).
   - Pade approximant: $\mathbf{\Gamma}_H \boldsymbol{f} = \begin{bmatrix} \mathbf{0}_N \\ \times \end{bmatrix}$.
   - Equation error: $\mathbf{\Gamma}_H \boldsymbol{f} = \begin{bmatrix} \mathbf{0}_N \\ \mathbf{0}_N \end{bmatrix}$.
2. $R(z) = 1$, then $f(z) = V(z)$ a unit norm function (output error methods).
   - Output error: Consider the minimization of $\|H(z) - \hat{H}(z)\|^2$ using orthogonal representation for $H(z) = \sum_{k=0}^{\infty} \tilde{h}_k F_k(z)$ ($\tilde{h}_k =< H(z), F_k(z) >$) and $\hat{H}(z) = \sum_{k=0}^{N} \nu_k F_k(z)$, then the optimal choice of $\nu_k = \tilde{h}_k =< H(z), F_k(z) >$ leads to the remaining error

$$\begin{aligned} H(z) - \hat{H}(z) &= \tilde{h}_{N+1}F_{N+1}(z) + \tilde{h}_{N+2}F_{N+2}(z) + \tilde{h}_{N+3}F_{N+3}(z) + ... \\ &= V(z)\sum_{k=1}^{\infty} \tilde{h}_{N+k} z^{-k} \end{aligned}$$

where $F_{N+k}(z) = z^{-k}V(z)$ was used.
Using the expansions $H(z) = \sum_{k=0}^{\infty} h_k z^{-k}$ and $V(z) = \sum_{k=0}^{\infty} v_k z^{-k}$, we can express

$$\tilde{h}_{N+k} =< H(z), z^{-k}V(z) >= h_k v_0 + h_{k+1}v_1 + h_{k+2}v_2 + ...$$

$$\begin{bmatrix} \tilde{h}_{N+1} \\ \tilde{h}_{N+2} \\ \tilde{h}_{N+3} \\ \vdots \end{bmatrix} = \mathbf{\Gamma}_H \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \end{bmatrix}$$

or $\sum_{k=1}^{\infty} \tilde{h}_{N+k} z^{-k} = [H(z)V(z^{-1})]_+$. Then the output error is

$$\|H(z) - \hat{H}(z)\|^2 = \|[H(z)V(z^{-1})]_+\|^2$$

In the general case $deg\ H(z) < N$ the best we can do is to force $V(z)$ to lie in an **approximate** null space of $\mathbf{\Gamma}_H$ in the sense that $\|\mathbf{\Gamma}_H \boldsymbol{v}\|$ is minimized.

4.4.4 Hankel norm rational approximation

Let $\mathbf{\Gamma}_H$ be approximated by $\mathbf{\Gamma}_{\hat{H}}$, then the approximation problem has a closed form solution with

$$\min_{rank\,\Gamma_{\hat{H}} \leq N} \|\mathbf{\Gamma}_H - \mathbf{\Gamma}_{\hat{H}}\| \;=\; \sigma_{N+1}(\mathbf{\Gamma}_H)$$

A physical interpretation of $\|\mathbf{\Gamma}_H\| = \sigma_1$ with rank equal N, for a balanced realization.

- Considering $\|\mathbf{\Gamma}_H\| = \max_{\|\boldsymbol{u}\|=1} \|\mathbf{\Gamma}_H \boldsymbol{u}\|$, $\boldsymbol{u} = [u(0)u(-1)u(-2)\cdots]^T$,then $\boldsymbol{y} = [y(1)\,y(2)\,y(3)\;\cdots]^T = \mathbf{\Gamma}_H \boldsymbol{u} = \mathcal{O}\mathcal{C}\boldsymbol{u}$.
- It is not hard to see that $\mathcal{C}\boldsymbol{u} = \boldsymbol{x}(1)$, where $\boldsymbol{x}(1)$ is the state vector subject to an initial condition having been produced by a unit norm vector,
- Using the singular value decomposition of $\mathbf{\Gamma}_H$

$$\mathbf{\Gamma}_H \;=\; [\eta_1\,\eta_2\;\cdots\;\eta_N]\begin{bmatrix}\sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_N\end{bmatrix}\begin{bmatrix}\xi_1^T \\ \xi_2^T \\ \vdots \\ \xi_N^T\end{bmatrix}$$

then $\boldsymbol{y} = [\eta_1\,\eta_2\;\cdots\;\eta_N]\begin{bmatrix}\sigma_1^{1/2} & & & \\ & \sigma_2^{1/2} & & \\ & & \ddots & \\ & & & \sigma_N^{1/2}\end{bmatrix}\begin{bmatrix}x_1(1) \\ x_2(1) \\ \vdots \\ x_N(1)\end{bmatrix}$
$= \sum_{k=1}^{N} \eta_k \sigma_k^{1/2} x_k(1)$ or, since $\{\eta_k\}$ are orthonormal $\|\boldsymbol{y}\|^2 = \sum_{k=1}^{N} \sigma_k [x_k(1)]^2$

Another important property

$$\|H(z) - \hat{H}(z)\|^2 \leq \sigma_1(\mathbf{\Gamma}_H - \mathbf{\Gamma}_{\hat{H}}) \leq sup_{|z|=1}|H(z) - \hat{H}(z)|$$

an upper bound for the approximation in $\mathcal{H}_2$.

Indeed, if we consider the *Frobenius norm*, i.e, that is defined for a matrix

$$\boldsymbol{P} = \begin{bmatrix} p_{1,1} & p_{1,2} & p_{1,3} & \cdots \\ p_{2,1} & p_{2,2} & p_{2,3} & \cdots \\ p_{3,1} & p_{3,2} & p_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

as $\|\boldsymbol{P}\|_F = \left(\sum_{k,l=1}^{\infty} p_{k,l}^2\right)^{1/2}$.

Lemma: Let $\boldsymbol{D} = diag[d_0, d_1, d_2, ...]$, where $d_k = d_{k-1}\sqrt{\frac{2k-1}{2k}}$, $d_0 = 1$. Then for any Hankel form $\boldsymbol{\Gamma}_H$,

$$\|\boldsymbol{D}\boldsymbol{\Gamma}_H\boldsymbol{D}\|_F = \left(\sum_{k=1}^{\infty} h_k^2\right)^{1/2} = \|[H(z)]_+\|^2$$

That leads to a priori lower bound

$$\begin{aligned} \min_{deg\ \hat{H}(z)=N} \|H(z) - \hat{H}(z)\|^2 &= \min_{rank\ \Gamma_{\hat{H}}=N} \|\boldsymbol{D}(\boldsymbol{\Gamma}_H - \boldsymbol{\Gamma}_{\hat{H}})\boldsymbol{D}\|_F \\ &\geq \sum_{k=N+1}^{\infty} \sigma_k^2(\boldsymbol{D}\boldsymbol{\Gamma}_H\boldsymbol{D}) \end{aligned}$$

4.5 Stability theory concepts

4.5.1 Stability of quasi-invariant systems

Convergence in the mean of an IIR adaptive algorithm can be studied by a related difference equation.

If the average behavior of the algorithm can be written as

$$E\{\tilde{\boldsymbol{\theta}}(n+1)\} = [\boldsymbol{R}_1 + \boldsymbol{R}_2(n)]E\{\tilde{\boldsymbol{\theta}}(n)\} + \boldsymbol{R}_3(n) \tag{44}$$

where $E\{\boldsymbol{\theta}(n) - \boldsymbol{\theta}_o\} = E\{\tilde{\boldsymbol{\theta}}(n)\}$ with $\boldsymbol{\theta}_o$ defining the ideal parameters, and $\boldsymbol{R}_1$ is positive definite and $\boldsymbol{R}_2(n)$ has norm sufficiently small, then this system is called **quasi-invariant**.

Theorem: Let the quasi-invariant system defined by equation (44), i.e.,

- $\boldsymbol{R}_1$ satisfy $\|\boldsymbol{R}_1^n\| < c\beta^n$ with c and β are constants such that $c > 0$ and $0 < \beta < 1$,
- $\boldsymbol{R}_2(n)$ has a norm sufficiently low, i.e., $\|\boldsymbol{R}_2(n)\| \leq \kappa_2$, for κ_2 a positive constant.
- $\boldsymbol{R}_3(n)$ has bounded norm, i.e., $\|\boldsymbol{R}_3(n)\| \leq \kappa_3$, for κ_3 a positive constant.

Then if $0 < (\beta + c\,\kappa_2) < 1$ the system of equation (44) is asymptotically stable.

Corollary: Note that if $\|\boldsymbol{R}_3(n)\|$ tends to zero for $n \to \infty$, then the system of equation (44) converge asymptotically to the origin, i.e.,

$$E\{\boldsymbol{\theta}(n+1)\} \to \boldsymbol{\theta}_o \tag{45}$$

for $n \to \infty$.

4.5.2 Stability of a non linear feedback system

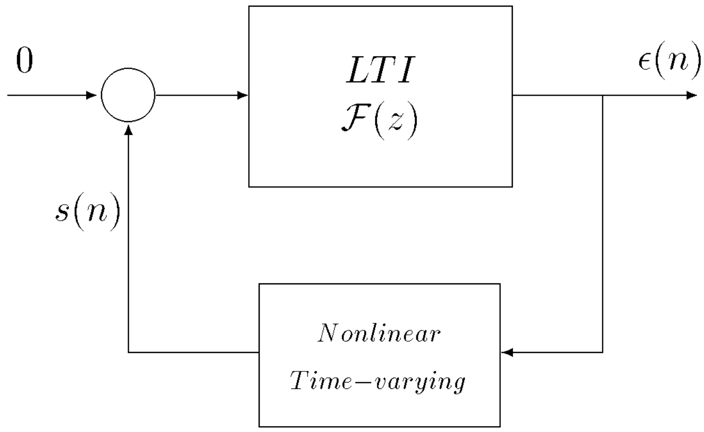

Figure 26: Nonlinear feedback system

Consider $\mathcal{F}(z)$ a rational transfer function and the feedback law related to the figure of the form (Popov inequality)

$$\sum_{n=0}^{N} s(n)\epsilon(n) \;\leq\; \gamma^2$$

Theorem: The closed-loop system of the figure is asymptotically stable (i.e., $s(n)$ and $\epsilon(n)$ remain bounded and tend to zero) for all feedback laws as the specified, and for all initial conditions, if and only if $\mathcal{F}(z)$ is strictly positive real, i.e., a stable and causal function such that: $Re\mathcal{F}(e^{jw}) \geq c \geq 0$ for all w.

Properties of positive real functions

- If $Re\mathcal{F}(e^{jw}) \geq c > 0$ for all w, then $Re\mathcal{F}(z) \geq c > 0$ for all $|z| \geq 1$.
- If $\mathcal{F}(z)$ is strictly positive real (SPR), then it can have no zeros in $|z| \geq 1$, i.e., if SPR then minimum phase (the converse is not true).
- If $\mathcal{F}(z)$ is SPR, so is its inverse $1/\mathcal{F}(z)$.
- Suppose $\mathcal{F}(z)$ SPR, and let $\epsilon(n) = \mathcal{F}(z)s(n)$. Then for all non zero square summable $\{s(n)\}$: $\sum_{k=-\infty}^{\infty} s(k)\epsilon(k) > 0$.

Proof of the Hyperstability theorem

Consider $u(n)$ and $y(n)$ such that

$$\begin{aligned} u(n) &= s(n) + \epsilon(n) = [\mathcal{F}(z) + 1]s(n) \\ y(n) &= s(n) - \epsilon(n) = [\mathcal{F}(z) - 1]s(n) \end{aligned}$$

that form $y(n) = \frac{\mathcal{F}(z)-1}{\mathcal{F}(z)+1}u(n) = \mathcal{G}(z)u(n)$, also a rational function with realization $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)$. If the bounded sequences $u(n)$ and $y(n)$ tend asymptotically to zero the same apply to $s(n)$ and $\epsilon(n)$.

From the properties of SPR functions, if $\mathcal{F}(z)$ is SPR, then $|\mathcal{G}(z)| \leq c < 1$ for all $|z| \geq 1$. Based on an bounded initial condition of the state vector of the system realizing $\mathcal{G}(z)$, $\boldsymbol{x}(0)$, that can be written as

$$\boldsymbol{x}(0) = [\boldsymbol{b}\ \boldsymbol{A}\boldsymbol{b}\ \boldsymbol{A}^2\boldsymbol{b} \ldots] \begin{bmatrix} u(-1) \\ u(-2) \\ u(-3) \\ \vdots \end{bmatrix}$$

and that for N time instant we can write

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} \boldsymbol{c}^T \\ \boldsymbol{c}^T\boldsymbol{A} \\ \vdots \\ \boldsymbol{c}^T\boldsymbol{A}^N \end{bmatrix} \boldsymbol{x}(0) + \begin{bmatrix} d & 0 & \cdots & 0 \\ \boldsymbol{c}^T\boldsymbol{b} & d & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \boldsymbol{c}^T\boldsymbol{A}^{N-1}\boldsymbol{b} & \cdots & \boldsymbol{c}^T\boldsymbol{b} & d \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N) \end{bmatrix}$$

Then is not hard to shown that (extending the use of Parseval theorem to a bounded initial condition)

$$\sum_{n=0}^{N} y^2(n) \leq c^2 \sum_{n=0}^{N} u^2(n) + f[\boldsymbol{x}(0)] \tag{46}$$

where $f[\boldsymbol{x}(0)]$ is a bounded function of the initial condition of the state vector of the system realizing $\mathcal{G}(z)$.

On the other hand, sustituting $s(n)$ and $\epsilon(n)$ in the Popov inequality we obtain

$$\sum_{n=0}^{N} u^2(n) \;\leq\; \sum_{n=0}^{N} y^2(n) + 4\gamma^2 \tag{47}$$

By using this in (46)

$$\begin{array}{rcl}\sum_{n=0}^{N} y^2(n) & \leq & c^2 \sum_{n=0}^{N} y^2(n) + 4c^2\gamma^2 + f[\boldsymbol{x}(0)] \\ & or & \\ \sum_{n=0}^{N} y^2(n) & \leq & \dfrac{4c^2\gamma^2 + f[\boldsymbol{x}(0)]}{1-c^2} < \infty\end{array}$$

this implies that $y^2(n) \to 0$ for $n \to \infty$ and by (47) the same can be inferred for $u^2(n)$.

Passive Impedance functions

Using $p = \frac{z-1}{z+1}$ and $\mathcal{F}(p)$ (a continuous time transfer function) an **impedance**. If $S(p)$ is the Laplace transform of a casual $s(t)$ electrical current, then $\epsilon(t)$ is the resulting voltage.

If $\mathcal{F}(p)$ is SPR,

$$\begin{aligned} \int_0^\infty s(t)\epsilon(t)dt &= \int_{-\infty}^{\infty} \mathcal{F}(j\Omega)\mathcal{S}(j\Omega)\mathcal{S}^*(j\Omega)d\Omega \\ &= \int_{-\infty}^{\infty} \mathcal{F}(j\Omega)|\mathcal{S}(j\Omega)|^2 d\Omega > 0 \end{aligned}$$

then the impedance is said to be **passive**.

Spectral factorization

Since for an stationary stochastic process with correlation $\{r_k\}$, with $r_k = r_{-k}$, $\mathcal{S}(z) = \sum_{k=-\infty}^{\infty} r_k z^{-k}$ is nonnegative along $|z| = 1$, then by chosen $\mathcal{F}(z) = r_0/2 + r_1 z^{-1} + r_2 z^{-2} + ...$, is easy to see that $\mathcal{F}(z^{-1}) + \mathcal{F}(z) = \mathcal{S}(z)$. Or, $\mathcal{F}(z)$ is SPR if and only if it is the (unilateral) z-transform of a correlation sequence $\{r_k\}$.

Also, if $\mathcal{S}(z)$ has positive geometric mean, i.e.,

$$exp\left(\frac{1}{2\pi}\int_{-\pi}^{pi} \log[\mathcal{S}(e^{jw})]\right) > 0$$

then it admits a **spectral factorization**: $\mathcal{S}(z) = F(z)F(z^{-1})$, for some stable and causal $F(z)$. The stochastic process which furnishes the correlation r_k could be modelled as the output of $F(z)$ driven by unit-variance white noise.

Positive real lemma: A rational function $\mathcal{F}(z) = d + \boldsymbol{c}(z\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}$ is positive real if and only if there exists a symmetric, positive definite $\boldsymbol{P}$ for which the symmetric matrix

$$\begin{bmatrix} \boldsymbol{P} - \boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A} & \boldsymbol{c} - \boldsymbol{A}^T\boldsymbol{P}\boldsymbol{b} \\ \boldsymbol{c}^T - \boldsymbol{b}^T\boldsymbol{P}\boldsymbol{A} & 2d - \boldsymbol{c}\boldsymbol{P}\boldsymbol{c} \end{bmatrix} = \begin{bmatrix} \boldsymbol{L}^T \\ \boldsymbol{N}^T \end{bmatrix} [\boldsymbol{L}\ \ \boldsymbol{N}]$$

is positive definite.

Then

$$F(z) = \boldsymbol{N} + \boldsymbol{L}(z\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}$$

5 MSOE minimization

MSOE minimization and related algorithms

- Stationary points (existence of local minima),
- ODE (convergence to local minima and instability).
- Direct-form realization of an adaptive IIR filter: implementation of the derivatives, simplifications.
- Lattice realization: simplifications.
- Other realizations