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THE FAST FOURIER TRANSFORM

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The Fourier transform has long been a principle analytical tool in such diverse fields as linear systems, optics, probability theory, quantum physics, antennas, and signal analysis. A similar statement is not true for the discrete Fourier transform. Even with the tremendous computing speeds available with modern computers, the discrete Fourier transform found relatively few applications because of the exorbitant amount of computation time required. However, with the development of the fast Fourier transform (an algorithm that efficiently computes the discrete Fourier transform), many facets of scientific analysis have been completely revolutionized.

As with any new development that brings about significant technological change, there is the problem of communicating the essential basics of the fast Fourier transform (FFT). A unified presentation which relates this technique to one's formal education and practical experience is dictated. The central aim of this book is to provide the student and the practicing professional a readable and meaningful treatment of the FFT and its basic application.

The book communicates with the reader not by the introduction of the topics but rather in the manner by which the topics are presented. Every major concept is developed by a three stage sequential process. First, the concept is introduced by an intuitive development which is usually pictorial in nature. Second, a non-sophisticated (but theoretically sound) mathematical treatment is developed to support the intuitive arguments. The third stage consists of practical examples designed to review and expand the concept being discussed. It is felt that this three step procedure gives meaning as well as mathematical substance to the basic properties of the FFT.

The book should serve equally well to senior or first year graduate stu-
PREFACE

dents and to the practicing scientific professional. As a text, the material covered can be easily introduced into course curriculums including linear systems, transform theory, systems analysis, signal processing, simulation, communication theory, optics, and numerical analysis. To the practicing engineer the book offers a readable introduction to the FFT as well as providing a unified reference. All major developments and computing procedures are tabulated for ease of reference.

Apart from an introductory chapter which introduces the Fourier transform concept and presents a historical review of the FFT, the book is essentially divided into four subject areas:

1. The Fourier Transform

In Chapters 2 through 6 we lay the foundation for the entire book. We investigate the Fourier transform, its inversion formula, and its basic properties; graphical explanations of each discussion lends physical insight to the concept. Because of their extreme importance in FFT applications the transform properties of the convolution and correlation integrals are explored in detail: Numerous examples are presented to aid in interpreting the concepts. For reference in later chapters the concept of Fourier series and waveform sampling are developed in terms of Fourier transform theory.

2. The Discrete Fourier Transform

Chapters 6 through 9 develop the discrete Fourier transform. A graphical presentation develops the discrete transform from the continuous Fourier transform. This graphical presentation is substantiated by a theoretical development. The relationship between the discrete and continuous Fourier transform is explored in detail: numerous waveform classes are considered by illustrative examples. Discrete convolution and correlation are defined and compared with continuous equivalents by illustrative examples. Following a discussion of discrete Fourier transform properties, a series of examples is used to illustrate techniques for applying the discrete Fourier transform.

3. The Fast Fourier Transform

In Chapters 10 through 12 we develop the FFT algorithm. A simplified explanation of why the FFT is efficient is presented. We follow with the development of a signal flow graph, a graphical procedure for examining the FFT. Based on this flow graph we describe sufficient generalities to develop a computer flow chart and FORTRAN and ALGOL computer programs. The remainder of this subject area is devoted toward theoretical development of the FFT algorithm in its various forms.

4. Basic Application of the FFT

Chapter 13 investigates the basic application of the FFT, computing discrete convolution and correlation integrals. In general, applications of
the FFT (systems analysis, digital filtering, simulation, power spectrum analysis, optics, communication theory, etc.) are based on a specific implementation of the discrete convolution or correlation integral. For this reason we describe in detail the procedures for applying the FFT to these discrete integrals.

A full set of problems chosen specifically to enhance and extend the presentation is included for all chapters.

I would like to take this opportunity to thank the many people who have contributed to this book. David E. Thouin, Jack R. Grisham, Kenneth W. Daniel, and Frank W. Goss assisted by reading various portions of the manuscript and offering constructive comments. Barry H. Rosenberg contributed the computer programs in Chapter 10 and W. A. J. Sippel was responsible for all computer results. Joanne Spiessbach compiled the bibliography. To each of these people I express my sincere appreciation.

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E. O. Brigham, Jr.