FOURIER TRANSFORM PROPERTIES

In dealing with Fourier transforms there are a few properties which are basic to a thorough understanding. A visual interpretation of these fundamental properties is of equal importance to knowledge of their mathematical relationships. The purpose of this chapter is to develop not only the theoretical concepts of the basic Fourier transform pairs, but also the meaning of these properties. For this reason we use ample analytical and graphical examples.

3-1 LINEARITY

If \( x(t) \) and \( y(t) \) have the Fourier transforms \( X(f) \) and \( Y(f) \), respectively, then the sum \( x(t) + y(t) \) has the Fourier transform \( X(f) + Y(f) \). This property is established as follows:

\[
\int x(t) e^{-j2\pi ft} \, dt + \int y(t) e^{-j2\pi ft} \, dt = \int (x(t) + y(t)) e^{-j2\pi ft} \, dt
\]

\( X(f) + Y(f) \) (3-1)

Fourier transform pair

\( x(t) + y(t) \quad \Box \quad X(f) + Y(f) \) (3-2)

is of considerable importance because it reflects the applicability of the Fourier transform to linear system analysis.

EXAMPLE 3-1

To illustrate the linearity property, consider the Fourier transform pairs

\( x(t) \quad \Box \quad X(f) \) (3-3)

\( K \quad \Box \quad K\delta(f) \)
\[ y(t) = A \cos(2\pi f_0 t) \quad \implies \quad X(f) = A \frac{\delta(f - f_0) + A}{2} \delta(f + f_0) \quad (3-4) \]

By the linearity theorem

\[ x(t) + y(t) = K + A \cos(2\pi f_0 t) \quad \implies \quad X(f) + Y(f) = K \delta(f) + A \frac{\delta(f - f_0) + A}{2} \delta(f + f_0) \]

\[ + A \frac{\delta(f + f_0)}{2} \quad (3-5) \]

Figures 3-1(a), (b), and (c), illustrate each of the Fourier transform pairs, respectively.

### 3-2 SYMMETRY

If \( h(t) \) and \( H(f) \) are a Fourier transform pair then

\[ h(t) \implies h(-f) \quad (3-6) \]

Fourier transform pair \((3-6)\) is established by rewriting Eq. \((2-5)\)

\[ h(-t) = \int_{-\infty}^{\infty} H(f) e^{-j2\pi ft} \, df \quad (3-7) \]

and by interchanging the parameters \( t \) and \( f \)

\[ h(-f) = \int_{-\infty}^{\infty} H(t) e^{-j2\pi ft} \, dt \quad (3-8) \]

**Example 3-2**

To illustrate this property consider the Fourier transform pair

\[ h(t) = A \quad |t| < T_0 \quad \implies \quad \frac{2AT_0 \sin(2\pi T_0 f)}{2\pi T_0 f} \quad (3-9) \]

illustrated previously in Fig. 2-3. By the symmetry theorem

\[ \frac{2AT_0 \sin(2\pi T_0 f)}{2\pi T_0 f} \implies h(-f) = h(f) = A \quad |f| < T_0 \quad (3-10) \]

which is identical to the Fourier transform pair \((2-17)\) illustrated in Fig. 2-5. Utilization of the symmetry theorem can eliminate many complicated mathematical developments; a case in point is the development of the Fourier transform pair \((2-27)\).

### 3-3 TIME SCALING

If the Fourier transform of \( h(t) \) is \( H(f) \), then the Fourier transform of \( h(kt) \) where \( k \) is a real constant greater than zero is determined by substituting \( t' = kt \) in the Fourier integral equation:

\[ \int_{-\infty}^{\infty} h(kt) e^{-j2\pi ft} \, dt \cdot \int_{-\infty}^{\infty} h(t') e^{-j2\pi ft'} \, dt' = \frac{1}{k} H\left(\frac{f}{k}\right) \quad (3-11) \]
For $k$ negative, the term on the right-hand side changes sign because the limits of integration are interchanged. Therefore, time scaling results in the Fourier transform pair
\[
\hat{h}(kt) \quad \frac{1}{|k|} H\left(\frac{f}{k}\right)
\]  
(3-12)
Figure 3-2. Time scaling property.
When dealing with time scaling of impulses, extra care must be exercised; from Eq. (A-10)

\[ \delta(at) = \frac{1}{|a|} \delta(t) \]  

(3-13)

**Example 3-3**

The time scaling Fourier transform property is well-known in many fields of scientific endeavor. As shown in Fig. 3-2 time scale expansion corresponds to frequency scale compression. Note that as the time scale expands, the frequency scale not only contracts but the amplitude increases vertically in such a way as to keep the area constant. This is a well-known concept in radar and antenna theory.

### 3-4 Frequency Scaling

If the inverse Fourier transform of \( H(f) \) is \( h(t) \), the inverse Fourier transform of \( H(kf); \ k \) a real constant; is given by the Fourier transform pair

\[ \frac{1}{|k|} \hat{h} \left( \frac{t}{k} \right) \quad \Box \quad H(kf) \]  

(3-14)

Relationship (3-14) is established by substituting \( f' = kf \) into the inversion formula:

\[ \int_{-\infty}^{\infty} H(kf)e^{j2\pi ft'}df' = \int_{-\infty}^{\infty} H(f')e^{j2\pi f(t/k)}df' = \frac{1}{|k|} \hat{h} \left( \frac{t}{k} \right) \]  

(3-15)

Frequency scaling of impulse functions is given by

\[ \delta(at) = \frac{1}{|a|} \delta(t) \]  

(3-16)

**Example 3-4**

Analogous to time scaling, frequency scale expansion results in a contraction of the time scale. This effect is illustrated in Fig. 3-3. Note that as the frequency scale expands, the amplitude of the time function increases. This is simply a reflection of the symmetry property (3-6) and the time scaling relationship (3-12).

**Example 3-5**

Many texts state Fourier transform pairs in terms of the radian frequency \( \omega \). For example, Papoulis [2, page 44] gives

\[ h(t) \quad \sum \delta(t - nT) \quad \Box \quad H(\omega) = \frac{2\pi}{T} \sum \delta(\omega - \frac{2\pi n}{T}) \]  

(3-17)

By the frequency scaling relationship (3-16) we know that

\[ \frac{2\pi}{T} \sum \delta \left( \frac{2\pi (f - \frac{n}{T})}{T} \right) = \frac{1}{T} \sum \delta \left( f - \frac{n}{T} \right) \]  

(3-18)
Figure 3-3. Frequency scaling property.
and (3-17) can be rewritten in terms of the frequency variable \( f \)

\[
h(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad \Leftrightarrow \quad H(f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) \tag{3-19}
\]

which is Eq. (2-40).

### 3-5 Time-Shifting

If \( h(t) \) is shifted by a constant \( t_0 \) then by substituting \( s = t - t_0 \) the Fourier transform becomes

\[
\int_{-\infty}^{\infty} h(t - t_0)e^{-j2\pi f s} \, dt = \int_{-\infty}^{\infty} h(s)e^{-j2\pi f (t - t_0)} \, ds
\]

\[
e^{-j2\pi f t_0} \int_{-\infty}^{\infty} h(s)e^{-j2\pi f s} \, ds
\]

\[
e^{-j2\pi f t_0} H(f) \tag{3-20}
\]

The time-shifted Fourier transform pair is

\[
h(t - t_0) \quad \Leftrightarrow \quad H(f)e^{-j2\pi f t_0} \tag{3-21}
\]

**Example 3-6**

A pictorial description of this pair is illustrated in Fig. 3-4. As shown, time-shifting results in a change in the phase angle \( \theta(f) = \tan^{-1}[I(f)/R(f)] \). Note that time-shifting does not alter the magnitude of the Fourier transform. This follows since

\[
H(f)e^{-j2\pi f t_0} = H(f)[\cos(2\pi f t_0) - j \sin(2\pi f t_0)]
\]

and hence the magnitude is given by

\[
|H(f)e^{-j2\pi f t_0}| = \sqrt{H^2(f)\cos^2(2\pi f t_0) + \sin^2(2\pi f t_0)} = \sqrt{H^2(f)} \tag{3-22}
\]

where \( H(f) \) has been assumed to be real for simplicity. These results are easily extended to the case of \( H(f) \), a complex function.

### 3-6 Frequency Shifting

If \( H(f) \) is shifted by a constant \( f_0 \), its inverse transform is multiplied by \( e^{j2\pi f t_0} \)

\[
h(t)e^{j2\pi f t_0} \quad \Leftrightarrow \quad H(f - f_0) \tag{3-23}
\]

This Fourier transform pair is established by substituting \( s = f - f_0 \) into the inverse Fourier transform-defining relationship

\[
\int_{-\infty}^{\infty} H(f - f_0)e^{j2\pi f t} \, df = \int_{-\infty}^{\infty} H(s)e^{j2\pi (f - f_0) s} \, ds
\]

\[
e^{j2\pi f_0 t} \int_{-\infty}^{\infty} H(s)e^{j2\pi f_0 s} \, ds
\]

\[-e^{j2\pi f_0 t} h(t) \tag{3-24}\]
Figure 3-4. Time shifting property.
EXAMPLE 3-7

To illustrate the effect of frequency-shifting let us assume that the frequency function $H(f)$ is real. For this case, frequency-shifting results in a multiplication of the time function $h(t)$ by a cosine whose frequency is determined by the frequency shift $f_0$ (Fig. 3-5). This process is commonly known as modulation.

Figure 3-5. Frequency shifting property.
3-7 ALTERNATE INVERSION FORMULA

The inversion formula (2-5) may also be written as

\[ h(t) = \left[ \int_{-\infty}^{\infty} H^*(f)e^{-j2\pi ft} \, df \right]^* \] (3-25)

where \( H^*(f) \) is the conjugate of \( H(f) \); that is, if \( H(f) = R(f) + jI(f) \) then \( H^*(f) = R(f) - jI(f) \). Relationship (3-25) is verified by simply performing the conjugation operations indicated.

\[
\begin{align*}
  h(t) &= \left[ \int_{-\infty}^{\infty} H^*(f)e^{-j2\pi ft} \, df \right]^* \\
        &= \left[ \int_{-\infty}^{\infty} R(f)e^{-j2\pi ft} \, df - j \int_{-\infty}^{\infty} I(f)e^{-j2\pi ft} \, df \right]^* \\
        &= \left[ \int_{-\infty}^{\infty} [R(f) \cos (2\pi ft) - I(f) \sin (2\pi ft)] \, df \\
        &\quad - j \int_{-\infty}^{\infty} [R(f) \sin (2\pi ft) + I(f) \cos (2\pi ft)] \, df \right]^* \\
        &= \int_{-\infty}^{\infty} [R(f) \cos (2\pi ft) - I(f) \sin (2\pi ft)] \, df \\
        &\quad + j \int_{-\infty}^{\infty} [R(f) \sin (2\pi ft) + I(f) \cos (2\pi ft)] \, df \\
        &= \int_{-\infty}^{\infty} [R(f) + jI(f)]\cos (2\pi ft) + j \sin (2\pi ft)] \, df \\
        &= \int_{-\infty}^{\infty} H(f)e^{j2\pi ft} \, df \quad (3-26)
\end{align*}
\]

The significance of the alternate inversion formula is that now both the Fourier transform and its inverse contain the common term \( e^{-j2\pi ft} \). This similarity will be of considerable importance in the development of fast Fourier transform computer programs.

3-8 EVEN FUNCTIONS

If \( h_\text{a}(t) \) is an even function, that is, \( h_\text{a}(t) = h_\text{a}(-t) \), then the Fourier transform of \( h_\text{a}(t) \) is an even function and is real;

\[ h_\text{a}(t) \quad \Leftrightarrow \quad R_\text{a}(f) = \int_{-\infty}^{\infty} h_\text{a}(t) \cos (2\pi ft) \, dt \] (3-27)

This pair is established by manipulating the defining relationships;

\[
\begin{align*}
  H(f) &= \int_{-\infty}^{\infty} h_\text{a}(t)e^{-j2\pi ft} \, dt \\
        &= \int_{-\infty}^{\infty} h_\text{a}(t) \cos (2\pi ft) \, dt - j \int_{-\infty}^{\infty} h_\text{a}(t) \sin (2\pi ft) \, dt \\
        &= \int_{-\infty}^{\infty} h_\text{a}(t) \cos (2\pi ft) \, dt = R_\text{a}(f) \quad (3-28)
\end{align*}
\]
The imaginary term is zero since the integrand is an odd function. Since \( \cos(2\pi ft) \) is an even function then \( h_r(t) \cos(2\pi ft) = h_r(t) \cos(2\pi(-f)t) \) and \( H_r(f) = H_r(-f) \); the frequency function is even. Similarly, if \( H(f) \) is given as a real and even frequency function, the inversion formula yields

\[
h(t) = \int_{-\infty}^{\infty} H(f)e^{j2\pi ft} df = \int_{-\infty}^{\infty} R_r(f)e^{j2\pi ft} df
\]

\[
= \int_{-\infty}^{\infty} R_r(f) \cos(2\pi ft) df + j \int_{-\infty}^{\infty} R_r(f) \sin(2\pi ft) df
\]

\[
= \int_{-\infty}^{\infty} R_r(f) \cos(2\pi ft) df = h_r(t)
\]

\[
(3-29)
\]

**Example 3-8**

As shown in Fig. 3-6 the Fourier transform of an even time function is a real and even frequency function; conversely, the inverse Fourier transform of a real and even frequency function is an even function of time.

![Figure 3-6. Fourier transform of an even function.](image)

**3-9 Odd Functions**

If \( h_0(t) = -h_0(-t) \), then \( h_0(t) \) is an odd function, and its Fourier transform is an odd and imaginary function,

\[
H(f) = \int_{-\infty}^{\infty} h_0(t)e^{-j2\pi ft} dt
\]

\[
= \int_{-\infty}^{\infty} h_0(t) \cos(2\pi ft) dt - j \int_{-\infty}^{\infty} h_0(t) \sin(2\pi ft) dt
\]

\[
= -j \int_{-\infty}^{\infty} h_0(t) \sin(2\pi ft) dt = j I_0(f)
\]

(3-30)

The real integral is zero since the multiplication of an odd and an even function is an odd function. Since \( \sin(2\pi ft) \) is an odd function, then \( h_0(t) \sin(2\pi ft) = -h_0(-t) \sin(2\pi(f)t) \) and \( H_0(f) = -H_0(-f) \); the frequency function is odd. For \( H(f) \) given as an odd and imaginary function, then

\[
h(t) = \int_{-\infty}^{\infty} H(f)e^{j2\pi ft} df = \int_{-\infty}^{\infty} I_0(f)e^{j2\pi ft} df
\]

\[
= j \int_{-\infty}^{\infty} I_0(f) \cos(2\pi ft) df + j \int_{-\infty}^{\infty} I_0(f) \sin(2\pi ft) df
\]

\[
= j \int_{-\infty}^{\infty} I_0(f) \sin(2\pi ft) df = h_0(t)
\]

(3-31)
and the resulting \( h_{o}(t) \) is an odd function. The Fourier transform pair is thus established:

\[
h_{o}(t) \quad \square \quad jH_{o}(f) := -j \int_{-\infty}^{\infty} h_{o}(t) \sin (2\pi ft) \, dt \tag{3-32}
\]

**Example 3-9**

An illustrative example of this transform pair is shown in Fig. 3-7. The function \( h(t) \) depicted is odd; therefore, the Fourier transform is an odd and imaginary function of frequency. If a frequency function is odd and imaginary then its inverse transform is an odd function of time.

![Figure 3-7. Fourier transform of an odd function.](image)

### 3-10 WAVEFORM DECOMPOSITION

An arbitrary function can always be decomposed or separated into the sum of an even and an odd function:

\[
h(t) = \frac{h(t)}{2} + \frac{h(-t)}{2}
\]

\[
= \left[ \frac{h(t)}{2} + \frac{h(-t)}{2} \right] + \left[ \frac{h(t)}{2} - \frac{h(-t)}{2} \right]
\]

\[
= h_{e}(t) + h_{o}(t) \tag{3-33}
\]

The terms in brackets satisfy the definition of an even and an odd function, respectively. From Eqs (3-27) and (3-32) the Fourier transform of (3-33) is

\[
H(f) = R(f) + jI(f) = H_{e}(f) + jH_{o}(f) \tag{3-34}
\]

where \( H_{e}(f) = R(f) \) and \( H_{o}(f) = jI(f) \). We will show in Chapter 10 that decomposition can be utilized to increase the speed of computation of the discrete Fourier transform.

**Example 3-10**

To demonstrate the concept of waveform decomposition consider the exponential function [Fig. 3-8(a)]

\[
h(t) = e^{-\alpha t} \quad t \geq 0 \tag{3-35}
\]
Following the developments leading to (3-33) we obtain
\[
h(t) = \left[ \frac{e^{-at}}{2} \right] + \left[ \frac{e^{-at}}{2} \right]
\]
\[
\times \left[ \frac{e^{at}}{2} \right]_{t=0} + \left[ \frac{e^{at}}{2} \right]_{t=0} - \left[ \frac{e^{at}}{2} \right]_{t=0} + \left[ \frac{e^{at}}{2} \right]_{t=0}
\]
\[
\times \left[ \frac{e^{-at}}{2} \right]_{t=0} + \left[ \frac{e^{-at}}{2} \right]_{t=0}
\]
\[
\Rightarrow \{ h_e(t) \} + \{ h_o(t) \}
\]  
(3-36)

Figures 3-8(b) and (c) illustrate the even and odd decomposition, respectively.

3-11 COMPLEX TIME FUNCTIONS

For ease of presentation we have to this point considered only real functions of time. The Fourier transform (2-1), the inversion integral (2-5), and
the Fourier transform properties hold for the case of $h(t)$, a complex function
of time. If

$$h(t) = h_r(t) + jh_i(t)$$

(3-37)

where $h_r(t)$ and $h_i(t)$ are respectively the real part and imaginary part of the
complex function $h(t)$, then the Fourier integral (2-1) becomes

$$H(f) = \int_{-\infty}^{\infty} [h_r(t) + jh_i(t)]e^{-j2\pi ft} dt$$

$$= \int_{-\infty}^{\infty} [h_r(t) \cos (2\pi ft) + h_i(t) \sin (2\pi ft)] dt$$

$$-j \int_{-\infty}^{\infty} [h_i(t) \sin (2\pi ft) - h_r(t) \cos (2\pi ft)] dt$$

$$= R(f) + jI(f)$$

(3-38)

Therefore

$$R(f) = \int_{-\infty}^{\infty} [h_r(t) \cos (2\pi ft) + h_i(t) \sin (2\pi ft)] dt$$

(3-39)

$$I(f) = - \int_{-\infty}^{\infty} [h_i(t) \sin (2\pi ft) - h_r(t) \cos (2\pi ft)] dt$$

(3-40)

Similarly the inversion formula (2-5) for complex functions yields

$$h_r(t) = \int_{-\infty}^{\infty} [R(f) \cos (2\pi ft) - I(f) \sin (2\pi ft)] df$$

(3-41)

$$h_i(t) = \int_{-\infty}^{\infty} [R(f) \sin (2\pi ft) + I(f) \cos (2\pi ft)] df$$

(3-42)

If $h(t)$ is real, then $h(t) = h_r(t)$, and the real and imaginary parts of the
Fourier transform are given by Eqs. (3-39) and (3-40), respectively,

$$R_r(f) = \int_{-\infty}^{\infty} h_r(t) \cos (2\pi ft) dt$$

(3-43)

$$I_r(f) = - \int_{-\infty}^{\infty} h_r(t) \sin (2\pi ft) dt$$

(3-44)

$R_r(f)$ is an even function, since $R_r(f) = R_r(-f)$. Similarly $I_r(-f) = -I_r(f)$
and $I_r(f)$ is odd.

For $h(t)$ purely imaginary, $h(t) = jh_i(t)$ and

$$R_i(f) = \int_{-\infty}^{\infty} h_i(t) \cos (2\pi ft) dt$$

(3-45)

$$I_i(f) = \int_{-\infty}^{\infty} h_i(t) \sin (2\pi ft) dt$$

(3-46)

$R_i(f)$ is an odd function and $I_i(f)$ is an even function. Table 3-1 lists various
complex time functions and their respective Fourier transforms.

**Example 3-11**

We can employ relationships (3-43), (3-44), (3-45), and (3-46) to simultaneously
determine the Fourier transform of two real functions. To illustrate this point,
Table 3-1: Properties of the Fourier Transform for Complex Functions

<table>
<thead>
<tr>
<th>Time domain $h(t)$</th>
<th>Frequency domain $H(f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real</td>
<td>Real part even</td>
</tr>
<tr>
<td></td>
<td>Imaginary part odd</td>
</tr>
<tr>
<td>Imaginary</td>
<td>Real part odd</td>
</tr>
<tr>
<td></td>
<td>Imaginary part even</td>
</tr>
<tr>
<td>Real even, imaginary odd</td>
<td>Real</td>
</tr>
<tr>
<td>Real odd, imaginary even</td>
<td>Imaginary</td>
</tr>
<tr>
<td>Real and even</td>
<td>Real and even</td>
</tr>
<tr>
<td>Real and odd</td>
<td>Imaginary and odd</td>
</tr>
<tr>
<td>Imaginary and even</td>
<td>Imaginary and even</td>
</tr>
<tr>
<td>Imaginary and odd</td>
<td>Real and odd</td>
</tr>
<tr>
<td>Complex and even</td>
<td>Complex and even</td>
</tr>
<tr>
<td>Complex and odd</td>
<td>Complex and odd</td>
</tr>
</tbody>
</table>

Recall the linearity property (3-2):

$$x(t) + y(t) \quad X(f) = X(f) + Y(f) \quad (3-47)$$

Let $x(t) = h(t)$ and $y(t) = jg(t)$ where both $h(t)$ and $g(t)$ are real functions. It follows that $X(f) = H(f)$ and $Y(f) = jG(f)$. Since $x(t)$ is real then from (3-43) and (3-44)

$$x(t) = h(t) \quad X(f) = H(f) = R_h(f) + jI_h(f) \quad (3-48)$$

Similarly, since $y(t)$ is imaginary then from (3-45) and (3-46)

$$y(t) = jg(t) \quad Y(f) = jG(f) = R_g(f) + jI_g(f) \quad (3-49)$$

Hence

$$h(t) + jg(t) \quad H(f) + jG(f) \quad (3-50)$$

where

$$H(f) = R_h(f) + jI_h(f) \quad (3-51)$$
$$G(f) = I_h(f) - jR_h(f) \quad (3-52)$$

Thus if

$$z(t) = h(t) + jg(t) \quad (3-53)$$

then the Fourier transform of $z(t)$ can be expressed as

$$Z(f) = R(f) + jI(f)$$

$$= \left[ \frac{R(f)}{2} + \frac{R(-f)}{2} \right] + \left[ \frac{R(f)}{2} - \frac{R(-f)}{2} \right]$$

$$+ j \left[ \frac{I(f)}{2} + \frac{I(-f)}{2} \right] + j \left[ \frac{I(f)}{2} - \frac{I(-f)}{2} \right] \quad (3-54)$$
and from (3-51) and (3-52)

\[ H(f) = \left[ \frac{R(f) + R(-f)}{2} \right] + j\left[ \frac{I(f) - I(-f)}{2} \right] \]  

\[ G(f) = \left[ \frac{I(f) + I(-f)}{2} \right] - j\left[ \frac{R(f) - R(-f)}{2} \right] \]  

Thus it is possible to separate the frequency function \( Z(f) \) into the Fourier transforms of \( h(t) \) and \( g(t) \), respectively. As will be demonstrated in Chapter 10, this technique can be used advantageously to increase the speed of computation of the discrete Fourier transform.

### 3-12 SUMMARY OF PROPERTIES

For future reference the basic properties of the Fourier transform are summarized in Table 3-2. These relationships will be of considerable importance throughout the remainder of this book.

**TABLE 3-2 PROPERTIES OF FOURIER TRANSFORMS**

<table>
<thead>
<tr>
<th>Time domain</th>
<th>Equation no.</th>
<th>Frequency domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear addition ( x(t) \pm y(t) )</td>
<td>(3-2)</td>
<td>Linear addition ( X(f) \pm Y(f) )</td>
</tr>
<tr>
<td>Symmetry ( H(t) )</td>
<td>(3-6)</td>
<td>Symmetry ( h(f) )</td>
</tr>
<tr>
<td>Time scaling ( h(kt) )</td>
<td>(3-12)</td>
<td>Inverse scale change  ( \frac{1}{k} H\left(\frac{f}{k}\right) )</td>
</tr>
<tr>
<td>Inverse scale change  ( \frac{1}{k} h\left(\frac{t}{k}\right) )</td>
<td>(3-14)</td>
<td>Frequency scaling ( H(kt) )</td>
</tr>
<tr>
<td>Time shifting ( h(t - t_0) )</td>
<td>(3-21)</td>
<td>Phase shift ( H(f)e^{j2\pi f t_0} )</td>
</tr>
<tr>
<td>Modulation ( h(t)e^{j2\pi f_0 t} )</td>
<td>(3-23)</td>
<td>Frequency shifting ( H(f - f_0) )</td>
</tr>
<tr>
<td>Even function ( h(t) )</td>
<td>(3-27)</td>
<td>Real function ( H(f) = R(f) )</td>
</tr>
<tr>
<td>Odd function ( h(t) )</td>
<td>(3-30)</td>
<td>Imaginary ( H(f) = jI(f) )</td>
</tr>
<tr>
<td>Real function ( h(t) = h_1(t) )</td>
<td>(3-43)</td>
<td>Real part even ( H(f) = R(f) )</td>
</tr>
<tr>
<td>Imaginary function ( h(t) = jh(t) )</td>
<td>(3-45)</td>
<td>Real part odd ( H(f) = R(f) + jI(f) )</td>
</tr>
</tbody>
</table>

\( R(f) \) and \( I(f) \) are respectively the real and imaginary parts of \( H(f) \).
PROBLEMS

3-1. Let:
\[
\begin{align*}
    h(t) &= \begin{cases} 
        A & |t| < 2 \\
        \frac{A}{2} & |t| = \pm 2 \\
        0 & |t| > 2 
    \end{cases} \\
    x(t) &= \begin{cases} 
        -A & |t| < 1 \\
        \frac{A}{2} & |t| = \pm 1 \\
        0 & |t| > 1 
    \end{cases}
\end{align*}
\]
Sketch \( h(t) \), \( x(t) \), and \( [h(t) - x(t)] \). Use Fourier transform pair (2-21) and the linearity theorem to find the Fourier transform of \([h(t) - x(t)]\).

3-2. Consider the functions \( h(t) \) illustrated in Fig. 3-9. Use the linearity property to derive the Fourier transform of \( h(t) \).

3-3. Use the symmetry theorem and the Fourier transform pairs of Fig. 2-11 to determine the Fourier transform of the following:
   a. \( h(t) = \frac{A^2 \sin^2 (2\pi T_c t)}{(\pi T_c)^2} \)
   b. \( h(t) = \frac{(a^2 + 4\pi^2 T^2)}{a} \)
   c. \( h(t) = \exp \left( -\frac{\pi^2 t^2}{\alpha} \right) \)
3-4. Derive the frequency scaling property from the time scaling property by means of the symmetry theorem.

3-5. Consider

\[ h(t) = \begin{cases} A^2 - \frac{A^2}{2T_0} |t| & |t| < 2T_0 \\ 0 & |t| > 2T_0 \end{cases} \]

Sketch the Fourier transform of \( h(2t) \), \( h(4t) \), and \( h(8t) \). (The Fourier transform of \( h(t) \) is given in Fig. 2-11.)

3-6. Derive the time scaling property for the case \( \kappa \) negative.

3-7. By means of the shifting theorem find the Fourier transform of the following functions:

a. \( h(t) = A \sin \left( \frac{2\pi f_0 (t - t_0)}{\pi (t - t_0)} \right) \)

b. \( h(t) = k \delta(t - t_0) \)

c. \( h(t) = \begin{cases} A^2 - \frac{A^2}{2T_0} |t - t_0| & |t - t_0| < 2T_0 \\ 0 & |t - t_0| > 2T_0 \end{cases} \)

3-8. Show that

\[ h(\alpha t - \beta) \quad \bigotimes \quad \frac{1}{|\alpha|} e^{j2\pi \alpha \beta} \ast H \left( \frac{f}{\alpha} \right) \]

3-9. Show that \( |H(f)| = |e^{-j2\pi f_0} H(f)| \); that is, the magnitude of a frequency function is independent of the time delay.

3-10. Find the inverse Fourier transform of the following functions by using the frequency shifting theorem:

a. \( H(f) = \frac{A \sin \left( \frac{2\pi f_0 (f - f_0)}{\pi (f - f_0)} \right)}{\pi (f - f_0)} \)

b. \( H(f) = \frac{\alpha^2}{\alpha^2 + 4\pi^2 (f - f_0)^2} \)

c. \( H(f) = \frac{A^2 \sin^2 \left( \frac{2\pi f_0 (f - f_0)}{\pi (f - f_0)} \right)}{\pi (f - f_0)^2} \)

3-11. Review the derivations leading to Eqs. (2-9), (2-13), (2-20), (2-25), (2-26), and (2-32). Note the mathematics which result are real for the Fourier transform of an even function.

3-12. Decompose and sketch the even and odd components of the following functions:

a. \( h(t) = \begin{cases} 1 & 1 < t < 2 \\ 0 & \text{otherwise} \end{cases} \)

b. \( h(t) = \begin{cases} 1 & 2 \leq (t - 2)^2 \\ 0 & \text{otherwise} \end{cases} \)

c. \( h(t) = \begin{cases} -t + 1 & 0 < t \leq 1 \\ 0 & \text{otherwise} \end{cases} \)

3-13. Prove each of the properties listed in Table 3-1.

3-14. If \( h(t) \) is real, show that \( |H(f)| \) is an even function.
3-15. By making a substitution of variable in Eq. (2-28) show that

\[ \int_{-\infty}^{\infty} x(t)\delta(\alpha t - t_0) \, dt = \frac{1}{a} x\left(\frac{t_0}{a}\right) \]

3-16. Prove the following Fourier transform pairs:
   a. \( \frac{dh(t)}{dt} \quad \square \quad j2\pi fH(f) \)
   b. \(-j2\pi f\delta(t) \quad \square \quad \frac{dH(f)}{df} \)

3-17. Use the derivative relationship of Problem 3-16(a) to find the Fourier transform of a pulse waveform given the Fourier transform of a triangular waveform.

REFERENCES
