CONVOLUTION AND CORRELATION

In the previous chapter we investigated those properties which are fundamental to the Fourier transform. However, there exists a class of Fourier transform relationships whose importance far outranks those previously considered. These properties are the convolution and correlation theorems which are to be discussed at length in this chapter.

4-1 CONVOLUTION INTEGRAL

Convolution of two functions is a significant physical concept in many diverse scientific fields. However, as in the case of many important mathematical relationships, the convolution integral does not readily unveil itself as to its true implications. To be more specific, the convolution integral is given by

\[ y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau) \, d\tau = x(t) \ast h(t) \quad (4-1) \]

Function \( y(t) \) is said to be the convolution of the functions \( x(t) \) and \( h(t) \). Note that it is extremely difficult to visualize the mathematical operation of Eq. (4-1). We will develop the true meaning of convolution by graphical analysis.

4-2 GRAPHICAL EVALUATION OF THE CONVOLUTION INTEGRAL

Let \( x(t) \) and \( h(t) \) be two time functions given by graphs as represented in Fig. 4-1(a) and (b), respectively. To evaluate Eq. (4-1), functions \( x(\tau) \) and \( h(t - \tau) \) are required. \( x(\tau) \) and \( h(\tau) \) are simply \( x(t) \) and \( h(t) \), respectively,
where the variable $t$ has been replaced by the variable $\tau$. $h(-\tau)$ is the image of $h(\tau)$ about the ordinate axis and $h(t - \tau)$ is simply the function $h(-\tau)$ shifted by the quantity $t$. Functions $x(\tau)$, $h(-\tau)$, and $h(t - \tau)$ are shown in Fig. 4-2. To compute the integral Eq. (4-1), it is necessary to multiply and integrate the functions $x(\tau)$ [Fig. 4-2(a)] and $h(t - \tau)$ [Fig. 4-2(c)] for each value of $t$ from $-\infty$ to $+\infty$. As illustrated in Figs. 4-3(a) and (b), this product is zero for the choice of the parameter $t = -t_i$. The product remains zero until $t$ is reduced to zero. As illustrated in Figs. 4-3(c) and (d), the product of $x(\tau)$ and $h(t_i + \tau)$ is the function emphasized by shading. The integral of this function is simply the shaded area beneath the curve. As $t$ is increased to $2t_i$ and further to $3t_i$, Figs. 4-3(d), (e), and (f) illustrate the relationships of the functions to be multiplied as well as the resulting integrations. For $t = 4t_i$, the product again becomes zero as shown by Figs. 4-3(f) and (h). This product remains zero for all $t$ greater than $4t_i$ [Figs. 4-3(g) and (h)]. If $t$ is allowed to be a continuum of values, then the convolution of $x(t)$ and $h(t)$ is the triangular function illustrated in Fig. 4-3(h).

The procedure described is a convenient graphical technique for evaluating convolution integrals. Summarizing the steps:

1. **Folding.** Take the mirror image of $h(\tau)$ about ordinate axis.
2. **Displacement.** Shift $h(-\tau)$ by the amount $t$.
3. **Multiplication.** Multiply the shifted function $h(t - \tau)$ by $x(\tau)$.
4. **Integration.** Area under the product of $h(t - \tau)$ and $x(\tau)$ is the value of the convolution at time $t$.

**Example 4-1**

To illustrate further the rules for graphical evaluation of the convolution integral, convolve the functions illustrated in Figs. 4-4(a) and (b). First, fold $h(\tau)$ to obtain $h(-\tau)$ as illustrated in Fig. 4-4(c). Next, displace or shift $h(-\tau)$ by the amount $t$ as shown in Fig. 4-4(d). Then, multiply $h(t - \tau)$ by $x(\tau)$ [Fig. 4-4(e)] and finally, integrate to obtain the convolution result for time $t$ [Fig. 4-4(f)].
Figure 4-2. Graphical description of folding operation.

Figure 4-3. Graphical example of convolution.
Figure 4-4. Convolution procedure: folding, displacement, multiplication, and integration.
The result illustrated in Fig. 4.4(f) can be determined directly from (4.1)

\[ y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau) \, d\tau = \int_{0}^{\infty} (e^{-t}) \, d\tau \]

\[ = e^{-t}(e^t - 1) = 1 - e^{-t} \quad (4.2) \]

Note that the general convolution integration limits of $-\infty$ to $+\infty$ become 0 to $t$ for Ex. 4.1. It is desired to develop a straightforward approach to find the correct integration limits. For Ex. 4.1, the lower non-zero value of the function $h(t - \tau) = e^{-(t-\tau)}$ is $-\infty$ and the lower non-zero value for $x(\tau)$ is 0. When we integrated, we chose the largest of these two values as our lower limit of integration. The upper non-zero value of $h(t - \tau)$ is $t$; the upper non-zero value of $x(\tau)$ is $\infty$. We choose the smallest of these two for our upper limit of integration.

A general rule for determining the limits of integration can then be stated as follows:

Given two functions with lower non-zero values of $L_1$ and $L_2$ and upper non-zero values of $U_1$ and $U_2$, choose the lower limit of integration as $\max [L_1, L_2]$ and the upper limit of integration as $\min [U_1, U_2]$.

It should be noted that the lower and upper non-zero values for the fixed function $x(\tau)$ do not change; however, the lower and upper non-zero values of the sliding function $h(t - \tau)$ change as $t$ changes. Thus, it is possible to have different limits of integration for different ranges of $t$. A graphical sketch similar to Fig. 4.4-4 is also an extremely valuable aid in choosing the correct limit of integration.

### 4.3 ALTERNATE FORM OF THE CONVOLUTION INTEGRAL

The above graphical illustration is but one of the possible interpretations of convolution. Equation (4.1) can also be written equivalently as

\[ y(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau) \, d\tau \quad (4.3) \]

That is, either $h(\tau)$ or $x(\tau)$ can be folded and shifted.

To see graphically that Eqs. (4.1) and (4.3) are equivalent, consider the functions illustrated in Fig. 4.5(a). It is desired to convolve these two functions. The series of graphs on the left in Fig. 4.5 illustrates the evaluation of Eq. (4.1); the graphs on the right illustrate the evaluation of Eq. (4.3). The previously defined steps of (1) Folding, (2) Displacement, (3) Multiplication, and (4) Integration are illustrated by Figs. 4.5(b), (c), (d), and (e), respectively. As indicated by Fig. 4.5(e), the convolution of $x(\tau)$ and $h(\tau)$ is the same irrespective of which function is chosen for folding and displacement.
Figure 4.5. Graphical example of convolution by Eqs. (4-1) and (4-3).
Example 4-2

Let

\[ h(t) = \begin{cases} 
  e^{-t} & t \geq 0 \\
  0 & t < 0 
\end{cases} \]  \hfill (4-4)

and

\[ x(t) = \begin{cases} 
  \sin t & 0 \leq t \leq \frac{\pi}{2} \\
  0 & \text{otherwise} 
\end{cases} \]  \hfill (4-5)

Find \( h(t) \ast x(t) \) using both Eqs. (4-1) and (4-3).

From (4-1)

\[
y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) \, d\tau 
\]

\[
y(t) = \begin{cases} 
  \int_{0}^{t} \sin \tau e^{-u} \, d\tau & 0 \leq t \leq \frac{\pi}{2} \\
  \int_{\frac{\pi}{2}}^{t} \sin \tau e^{-u} \, d\tau & t \geq \frac{\pi}{2} \\
  0 & t \leq 0 
\end{cases} \]  \hfill (4-6)

The integral limits are easily determined by using the procedure described previously. The lower and upper non-zero value of the function \( x(\tau) \) is 0 and \( \pi/2 \), respectively. For the function \( h(t-\tau) = e^{-u} \) the lower non-zero value is \( -\infty \) and the upper non-zero value is \( t \). We take the maximum of the lower non-zero values for our lower limit of integration; i.e., 0. The upper limit of integration is a function of \( t \). For \( 0 \leq t \leq \pi/2 \) the minimum of the upper non-zero values is \( t \) and hence the upper limit of integration is \( t \). For \( t \geq \pi/2 \) the minimum of the upper non-zero values is \( \pi/2 \) and consequently the upper limit of integration for this range of \( t \) is \( \pi/2 \). A graphical sketch of the convolution process will also yield these integration limits.

Evaluating (4-6) we obtain

\[
y(t) = \begin{cases} 
  0 & t \leq 0 \\
  \frac{1}{2} (\sin t - \cos t + e^{-t}) & 0 < t \leq \frac{\pi}{2} \\
  e^{-t} (1 + e^{2}) & t \geq \frac{\pi}{2} 
\end{cases} \]  \hfill (4-7)

Similarly from (4-3) we obtain

\[
y(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) \, d\tau 
\]

\[
y(t) = \begin{cases} 
  \int_{0}^{t} e^{-i} \sin (t-\tau) \, d\tau & 0 < t < \frac{\pi}{2} \\
  \int_{t-\pi/2}^{t} e^{-i} \sin (t-\tau) \, d\tau & t \geq \frac{\pi}{2} \\
  0 & t < 0 
\end{cases} \]  \hfill (4-8)
Although Eqs. (4-8) are different from Eqs. (4-6), evaluation yields identical results to (4-7).

**4-4 CONVOLUTION INVOLVING IMPULSE FUNCTIONS**

The simplest type of convolution integral to evaluate is one in which either \( x(t) \) or \( h(t) \) is an impulse function. To illustrate this point, let \( h(t) \) be

\[ h(t) \]

\[ x(t) \]

\[ h(t) \times x(t) \]

*Figure 4-6. Illustration of convolution involving impulse functions.*
the singular function shown graphically in Fig. 4-6(a) and let \( x(t) \) be the
rectangular function shown in Fig. 4-6(b). For these example functions Eq. (4-1) becomes
\[
y(t) = \int_{-\infty}^{\infty} [\delta(t - T) + \delta(t + T)]x(t - \tau) d\tau
\]
(4-9)
Recall from Eq. (2-28) that
\[
\int_{-\infty}^{\infty} \delta(t - T) x(\tau) d\tau = x(T)
\]
Hence, Eq. (4-9) can be written as
\[
y(t) = x(t - T) + x(t + T)
\]
(4-10)
Function \( y(t) \) is illustrated in Fig. 4-6(c). Note that convolution of the function \( x(t) \) with an impulse function is evaluated by simply reconstructing \( x(t) \)
with the position of the impulse function replacing the ordinate of \( x(t) \). As we will see in the developments to follow, the ability to visualize convolution involving impulse function is of considerable importance.

**Example 4-3**

Let \( h(t) \) be a series of impulse functions as illustrated in Fig. 4-7(a). To evaluate the convolution of \( h(t) \) with the rectangular pulse shown in Fig. 4-7(b), we simply reproduce the rectangular pulse at each of the impulse functions. The resulting convolution results are illustrated in Fig. 4-7(c).

### 4-5 Convolution Theorem

Possibly the most important and powerful tool in modern scientific analysis is the relationship between Eq. (4-1) and its Fourier transform. This relationship, known as the convolution theorem, allows one the complete freedom to convolve mathematically (or visually) in the time domain by simple multiplication in the frequency domain. That is, if \( h(t) \) has the Fourier transform \( H(f) \) and \( x(t) \) has the Fourier transform \( X(f) \), then \( h(t) * x(t) \) has the Fourier transform \( H(f)X(f) \). The convolution theorem is thus given by the Fourier transform pair
\[
h(t) * x(t) \quad \overset{\square}{=} \quad H(f)X(f)
\]
(4-11)
To establish this result, first form the Fourier transform of both sides of Eq. (4-1)
\[
Y(f) = \int_{-\infty}^{\infty} x(\tau)[\int_{-\infty}^{\infty} h(t - \tau)e^{-j2\pi ft} dt]d\tau
\]
(4-12)
which is equivalent to (assuming the order of integration can be changed)
\[
Y(f) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)[h(t - \tau)e^{-j2\pi ft} dt]d\tau
\]
(4-13)
By substituting $\sigma = t - \tau$ the term in the brackets becomes

$$\int_{-\infty}^{\infty} h(\sigma) e^{-j2\pi f(\sigma + \tau)} d\sigma = e^{-j2\pi f\tau} \int_{-\infty}^{\infty} h(\sigma) e^{-j2\pi f\sigma} d\sigma$$

$$= e^{-j2\pi f\tau} H(f)$$  \hspace{1cm} (4-14)

Equation (4-13) can then be rewritten as

$$Y(f) = \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f\tau} H(f) d\tau = H(f)X(f)$$ \hspace{1cm} (4-15)

The converse is proven similarly.

**Example 4-4**

To illustrate the application of the convolution theorem, consider the convolution of the two rectangular functions shown in Figs. 4-8(a) and (b). As we have seen previously, the convolution of two rectangular functions is a triangular function as shown in Fig. 4-8(e). Recall from Fourier transform pair (2-21) that the Fourier transform of a rectangular function is the sinc $(f)/f$ function illustrated in Figs. 4-8(c) and (d). The convolution theorem states that convolution in the time domain

\[ y(t) = x(t) * h(t) \]

is equivalent to multiplication in the frequency domain

\[ Y(f) = X(f)H(f) \]
Figure 4-8. Graphical example of the convolution theorem.
corresponds to multiplication in the frequency domain; therefore, the triangular waveform of Fig. 4-8(e) and the sin^2 f/f^2 function of Fig. 4-8(f) are Fourier transform pairs. Thus, we can use the theorem as a convenient tool for developing additional Fourier transform pairs.

**Example 4-5**

One of the most significant contributions of distribution theory results from the fact that the product of a continuous function and an impulse function is well defined (Appendix A); hence, if \( h(t) \) is continuous at \( t = t_0 \) then

\[
h(t_0) \delta(t - t_0) = h(t_0) \delta(t - t_0)
\]  
(4-16)

This result coupled with the convolution theorem allows one to eliminate the tedious derivation of many Fourier transform pairs. To illustrate, consider the two time functions \( h(t) \) and \( x(t) \) shown in Figs. 4-9(a) and (b). As described previously, the convolution of these two functions is the infinite pulse train illustrated in Fig. 4-9(c). It is desired to determine the Fourier transform of this infinite sequence of pulses. We simply use the convolution theorem; the Fourier transform of \( h(t) \) is the sequence of impulse functions; transform pair (2-40), illustrated in Fig. 4-9(c), and the Fourier transform of a rectangular function is the \( \sin(f)/f \) function shown in Fig. 4-9(d). Multiplication of these two frequency functions yields the desired Fourier transform. As illustrated in Fig. 4-9(f), the Fourier transform of a pulse train is a sequence of impulse functions whose amplitude is weighted by a \( \sin(f)/f \) function. This is a well-known result in the field of radar systems. It is to be noted that the multiplication of the two frequency functions must be interpreted in the sense of distribution theory; otherwise the product is meaningless. We can see that the ability to change from a convolution in the time domain to multiplication in the frequency domain often renders unwieldy problems rather straightforward.

**4-6 FREQUENCY CONVOLUTION THEOREM**

We can equivalently go from convolution in the frequency domain to multiplication in the time domain by using the frequency convolution theorem; the Fourier transform of the product \( h(t)x(t) \) is equal to the convolution \( H(f) \ast X(f) \). The frequency convolution theorem is

\[
h(t)x(t) \quad \Longleftrightarrow \quad H(f) \ast X(f)
\]  
(4-17)

This pair is established by simply substituting the Fourier transform pair (4-11) into the symmetry Fourier transform relationship (3-6).

**Example 4-6**

To illustrate the frequency convolution theorem, consider the cosine waveform of Fig. 4-10(a) and the rectangular waveform of Fig. 4-10(b). It is desired to determine the Fourier transform of the product of these two functions [Fig. 4-10(e)]. The Fourier transforms of the cosine and rectangular waveforms are given in Figs.
Figure 4-9. Example application of the convolution theorem.
Figure 4-10. Graphical example of the frequency convolution theorem.
4-10(c) and (d), respectively. Convolution of these two frequency functions yields the function shown in Fig. 4-10(f); Figs. 4-10(e) and (f) are thus Fourier transform pairs. This is the well-known Fourier transform pair of a single, frequency modulated pulse.

4-7 PROOF OF PARSEVAL’S THEOREM

Because the convolution theorem often simplifies difficult problems, it is fitting to summarize this discussion by utilizing the theorem to develop a simple proof of Parseval’s Theorem. Consider the function \( y(t) = h(t) \cdot h(t) \). By the convolution theorem the Fourier transform of \( y(t) \) is \( H(f) \ast H(f) \); that is

\[
\int_{-\infty}^{\infty} h(t) e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} H(f) \cdot H(\sigma - f) df \tag{4-18}
\]

Setting \( \sigma = 0 \) in the above expression yields

\[
\int_{-\infty}^{\infty} h(t) dt = \int_{-\infty}^{\infty} H(f) \cdot H(-f) df = \int_{-\infty}^{\infty} |H(f)|^2 df \tag{4-19}
\]

The last equality follows since \( H(f) = R(f) + jI(f) \) and thus \( H(-f) = R(-f) + jI(-f) \). From Eqs. (3-43) and (3-44), \( R(f) \) is even and \( I(f) \) is odd. Consequently \( R(-f) = R(f) \); \( I(-f) = -I(f) \); and \( H(-f) = R(f) - jI(f) \). The product \( H(f) \cdot H(-f) \) is equal to \( R^2(f) + I^2(f) \) which is the square of the Fourier spectrum \( |H(f)| \) defined in Eq. (2-2). Equation (4-19) is Parseval’s Theorem; it states that the energy in a waveform \( h(t) \) computed in the time domain must equal the energy of \( H(f) \) as computed in the frequency domain. As shown the use of the convolution theorem allows us to prove rather simply an important result. The convolution theorem is fundamental to many facets of Fourier transform analysis and, as we shall see, the theorem is of considerable importance in the application of the fast Fourier transform.

4-8 CORRELATION

Another integral equation of importance in both theoretical and practical application is the correlation integral;

\[
z(t) = \int_{-\infty}^{\infty} x(\tau) h(t + \tau) d\tau \tag{4-20}
\]

A comparison of the above expression and the convolution integral (4-1) indicates that the two are closely related. The nature of this relationship is best described by the graphical illustrations of Fig. 4-11. The functions to be both convolved and correlated are shown in Fig. 4-11(a). Illustrations on the left depict the process of convolution as described in the previous section; illustrations on the right graphically portray the process of correlation. As
Figure 4.11. Graphical comparison of convolution and correlation.
evidenced in Fig. 4-11(b), the two integrals differ in that there is no folding of one of the integrands in convolution. The previously described rules of displacement, multiplication, and integration are performed identically for both convolution and correlation. For the special case where either \(x(t)\) or \(h(t)\) is an even function, convolution and correlation are equivalent; this follows since an even function and its image are identical and, thus, folding can be eliminated from the steps in computing the convolution integral.

**Example 4-7**

Correlate graphically and analytically the waveforms illustrated in Fig. 4-12(a).

According to the rules for correlation we displace \(h(t)\) by the shift \(\tau\); multiply by \(x(\tau)\); and integrate the product \(x(\tau) h(t + \tau)\) as illustrated in Figs. 4-12(b), (c), and (d), respectively.

From Eq. (4-20), for positive displacement \(\tau\) we obtain

\[
\begin{align*}
z(t) &= \int_{-\infty}^{\infty} x(\tau) h(t + \tau) \, d\tau \\
&= \int_{0}^{\infty} (1) \frac{Q}{a} \tau \, d\tau \\
&= \frac{Q}{2a} \left[ \frac{\tau^2}{2} \right]_0^{\infty} = \frac{Q}{2a} (a - t)^2 \quad 0 \leq t \leq a 
\end{align*}
\]

(4-21)

For negative displacement, see Fig. 4-12(c) to justify the limits of integration.

\[
\begin{align*}
z(t) &= \int_{-\infty}^{0} (1) \frac{Q}{a} \tau \, d\tau \\
&= \frac{Q}{2a} (a - t^2) \quad -a \leq t \leq 0 
\end{align*}
\]

(4-22)

A general rule can be developed for determining the limits of integration for the correlation integral (see Problem 4-15).

### 4.9 Correlation Theorem

Recall that convolution-multiplication forms a Fourier transform pair. A similar result can be obtained for correlation. To derive this relationship, first evaluate the Fourier transform of Eq. (4-20)

\[
\int_{-\infty}^{\infty} z(t) e^{-j2\pi f t} \, dt = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} x(\tau) h(t + \tau) \, d\tau \right] e^{-j2\pi f t} \, dt 
\]

(4-23)

or (assuming the order of integration can be interchanged)

\[
\mathcal{Z}(f) = \int_{-\infty}^{\infty} x(\tau) \left[ \int_{-\infty}^{\infty} h(t + \tau) \, e^{-j2\pi f t} \, dt \right] d\tau
\]

(4-24)
Figure 4.12. Correlation procedure: displacement, multiplication, and integration.
Let \( \sigma = t + \tau \) and rewrite the term in brackets as
\[
\int_{-\infty}^{\infty} h(\sigma) e^{-j2\pi f(\sigma-t)} d\sigma = e^{j2\pi ft} \int_{-\infty}^{\infty} h(\sigma) e^{-j2\pi f\tau} d\sigma
\]
\[= e^{j2\pi ft} H(f)\] (4-25)

Equation (4-24) then becomes
\[
Z(f) = \int_{-\infty}^{\infty} x(\tau) e^{j2\pi f\tau} H(f) d\tau
\]
\[= H(f) \left[ \int_{-\infty}^{\infty} x(\tau) \cos (2\pi f\tau) d\tau + j \int_{-\infty}^{\infty} x(\tau) \sin (2\pi f\tau) d\tau \right]
\]
\[= H(f)[R(f) + jI(f)]\] (4-26)

Now the Fourier transform of \( x(\tau) \) is given by
\[
X(f) = \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f\tau} d\tau
\]
\[= \int_{-\infty}^{\infty} x(\tau) \cos (2\pi f\tau) d\tau - j \int_{-\infty}^{\infty} x(\tau) \sin (2\pi f\tau) d\tau
\]
\[= R(f) - jI(f)\] (4-27)

The bracketed term of (4-26) and the expression on the right in (4-27) are called conjugates [defined in Eq. (3-25)]. Equation (4-26) may be written as
\[
Z(f) = H(f)X^*(f)
\] (4-28)

and the Fourier transform pair for correlation is
\[
\int_{-\infty}^{\infty} h(\tau) x(t + \tau) d\tau \quad \quad H(f)X^*(f)
\] (4-29)

Note that if \( x(t) \) is an even function then \( X(f) \) is purely real and \( X(f) = X^*(f) \). For these conditions the Fourier transform of the correlation integral is \( H(f)X(f) \) which is identical to the Fourier transform of the convolution integral. These arguments for identity of the two integrals are simply the frequency domain equivalents of the previously discussed time domain requirement for equality of the two integrals.

If \( x(t) \) and \( h(t) \) are the same function, Eq. (4-20) is normally termed the autocorrelation function; if \( x(t) \) and \( h(t) \) differ, the term crosscorrelation is normally used.

**Example 4-8**

Determine the autocorrelation function of the waveform
\[
h(t) = e^{-|t|} \quad t > 0
\]
\[= 0 \quad t < 0
\] (4-30)
From (4-20)
\[ z(t) = \int_{-\infty}^{\infty} h(\tau) h(t - \tau) d\tau \]
\[ = \int_{0}^{\infty} e^{-\alpha \tau} e^{-\frac{\alpha (t - \tau)}{2a}} d\tau \quad t > 0 \]
\[ = \int_{0}^{\infty} e^{-\beta (t - \tau)} e^{-\frac{\alpha (t - \tau)}{2a}} d\tau \quad t < 0 \]
\[ = \frac{e^{-\alpha|t|}}{2a} \quad -\infty < t < \infty \quad (4-31) \]

**PROBLEMS**

4-1. Prove the following convolution properties:
   a. Convolution is commutative: \( (h(t) \ast x(t)) = (x(t) \ast h(t)) \)
   b. Convolution is associative: \( h(t) \ast [g(t) \ast x(t)] = [h(t) \ast g(t)] \ast x(t) \)
   c. Convolution is distributive over addition: \( h(t) \ast [g(t) + x(t)] = h(t) \ast g(t) + h(t) \ast x(t) \)

4-2. Determine \( h(t) \ast g(t) \)
   a. \( h(t) = e^{-\alpha t} \quad t > 0 \)
      \[ = 0 \quad t < 0 \]
      \[ g(t) = e^{-\beta t} \quad t > 0 \]
      \[ = 0 \quad t < 0 \]
   b. \( h(t) = te^{-t} \quad t \geq 0 \)
      \[ = 0 \quad t < 0 \]
      \[ g(t) = e^{-t} \quad t > 0 \]
      \[ = 0 \quad t < 0 \]
   c. \( h(t) = te^{-t} \quad t \geq 0 \)
      \[ = 0 \quad t < 0 \]
      \[ g(t) = e^t \quad t < -1 \]
      \[ = 0 \quad t > -1 \]
   d. \( h(t) = 2e^{\alpha t} \quad t > 1 \)
      \[ = 0 \quad t < 0 \]
      \[ g(t) = 2e^{\beta t} \quad t < 0 \]
      \[ = 0 \quad t > 0 \]
   e. \( h(t) = \sin(2\pi t) \quad 0 \leq t \leq \frac{1}{2} \)
      \[ = 0 \quad \text{elsewhere} \]
      \[ g(t) = 1 \quad 0 < t < \frac{1}{8} \]
      \[ = 0 \quad t < 0; t > \frac{1}{8} \]
   f. \( h(t) = 1 - t \quad 0 < t < 1 \)
      \[ = 0 \quad t < 0; t > 1 \]
      \[ g(t) = h(t) \]
Figure 4-13.
g. \( h(t) = (a - |t|)^3 \quad -a \leq t \leq a \)
   \[= 0 \quad \text{elsewhere} \]

h. \( g(t) = e^{-at} \quad t > 0 \)
   \[= 0 \quad t < 0 \]

\( g(t) = 1 - t \quad 0 < t < 1 \)
   \[= 0 \quad t < 0; \ t > 1 \]

4-3. Graphically sketch the convolution of the functions \( x(t) \) and \( h(t) \) illustrated in Fig. 4-13.

4-4. Sketch the convolution of the two odd functions \( x(t) \) and \( h(t) \) illustrated in Fig. 4-14. Show that the convolution of two odd functions is an even function.

![Convolution Graph](image)

**Figure 4-14.**

4-5. Use the convolution theorem to graphically determine the Fourier transform of the functions illustrated in Fig. 4-15.

4-6. Analytically determine the Fourier transform of \( e^{-x^2} \ast e^{-y^2} \). (Hint: Use the convolution theorem.)

4-7. Use the frequency convolution theorem to graphically determine the convolution of the functions \( x(t) \) and \( h(t) \) illustrated in Fig. 4-16.

4-8. Graphically determine the correlation of the functions \( x(t) \) and \( h(t) \) illustrated in Fig. 4-13.

4-9. Let \( h(t) \) be a time-limited function which is non-zero over the range

\[-T_0/2 \leq t \leq T_0/2\]

Show that \( h(t) \ast h(t) \) is non-zero over the range \(-T_0 \leq t \leq T_0\); that is, \( h(t) \ast h(t) \) has a "width" twice that of \( h(t) \).

4-10. Show that if \( h(t) = f(t) \ast g(t) \) then

\[
\frac{dh(t)}{dt} = \frac{df(t)}{dt} \ast g(t) \ast f(t) \ast \frac{dg(t)}{dt}
\]

4-11. If \([x(t)]^3\) implies \([x(t) \ast x(t) \ast x(t)]\) how does one evaluate \([x(t)]_{1/2}\)?
Figure 4-15.
4-12. By means of the frequency convolution theorem, graphically determine the Fourier transform of the half-wave rectified waveform shown in Fig. 4-17(a). Using this result incorporate the shifting theorem to determine the Fourier transform of the full-wave rectified waveform shown in Fig. 4-17(b).

4-13. Graphically find the Fourier transform of the following functions:
   a. \( h(t) = A \cos^2(2\pi f_0 t) \)
   b. \( h(t) = A \sin^2(2\pi f_0 t) \)
   c. \( h(t) = A \cos^2(2\pi f_0 t) + A \cos^2(\pi f_0 t) \)

4-14. Find graphically the inverse Fourier transform of the following functions:
   a. \( \frac{\sin(2\pi f)}{(2\pi f)^2} \)
   b. \( \frac{1}{(1 + (2\pi f)^2)} \)
4-15. Develop a set of rules for determining the limits of integration for the correlation integral.

REFERENCES


