FOURIER SERIES
AND SAMPLED WAVEFORMS

In the technical literature, Fourier series is normally developed independently of the Fourier integral. However, with the introduction of distribution theory, Fourier series can be theoretically derived as a special case of the Fourier integral. This approach is significant in that it is fundamental in considering the discrete Fourier transform as a special case of the Fourier integral. Also fundamental to an understanding of the discrete Fourier transform is the Fourier transform of sampled waveforms. In this chapter we relate both of these relationships to the Fourier transform and thereby provide the framework for the development of the discrete Fourier transform in Chapter 6.

5-1 FOURIER SERIES

A periodic function \( y(t) \) with period \( T_0 \) expressed as a Fourier series is given by the expression

\[
y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos(2\pi nf_0 t) + b_n \sin(2\pi nf_0 t) \right]
\]  

(5-1)

where \( f_0 \) is the fundamental frequency equal to \( 1/T_0 \). The magnitude of the sinusoids or coefficients are given by the integrals

\[
a_n = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} y(t) \cos(2\pi nf_0 t) \, dt \quad n = 0, 1, 2, 3, \ldots
\]

(5-2)

\[
b_n = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} y(t) \sin(2\pi nf_0 t) \, dt \quad n = 1, 2, 3, \ldots
\]

(5-3)
By applying the identities

$$\cos(2\pi nf_{ot}) = \frac{1}{2}(e^{j2\pi nf_{ot}} + e^{-j2\pi nf_{ot}})$$  
(5.4)

and

$$\sin(2\pi nf_{ot}) = \frac{1}{2j}(e^{j2\pi nf_{ot}} - e^{-j2\pi nf_{ot}})$$  
(5.5)

expression (5-1) may be written as

$$y(t) = \frac{a_0}{2} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n - jb_n) e^{j2\pi nf_{ot}} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n + jb_n) e^{-j2\pi nf_{ot}}$$  
(5.6)

To simplify this expression, negative values of $n$ are introduced in Eqs. (5-2) and (5-3).

$$a_{-n} = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} y(t) \cos(-2\pi nf_{ot}) \, dt$$

$$= \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} y(t) \cos(2\pi nf_{ot}) \, dt$$

$$= a_n \quad n = 1, 2, 3, \ldots$$  
(5.7)

$$b_{-n} = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} y(t) \sin(-2\pi nf_{ot}) \, dt$$

$$= -\frac{2}{T_0} \int_{-T_0/2}^{T_0/2} y(t) \sin(2\pi nf_{ot}) \, dt$$

$$= -b_n \quad n = 1, 2, 3, \ldots$$  
(5.8)

Hence we can write

$$\sum_{n=1}^{\infty} a_n e^{j2\pi nf_{ot}} = \sum_{n=1}^{\infty} a_n e^{j2\pi nf_{ot}}$$  
(5.9)

and

$$\sum_{n=1}^{\infty} jb_n e^{-j2\pi nf_{ot}} = -\sum_{n=1}^{\infty} jb_n e^{-j2\pi nf_{ot}}$$  
(5.10)

Substitution of (5.9) and (5.10) into Eq. (5-6) yields

$$y(t) = \frac{a_0}{2} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n - jb_n) e^{j2\pi nf_{ot}}$$

$$= \sum_{n=1}^{\infty} \alpha_n e^{j2\pi nf_{ot}}$$  
(5.11)

Equation (5-11) is the Fourier series expressed in exponential form; coefficients $\alpha_n$ are, in general, complex. Since

$$\alpha_n = \frac{1}{2} (a_n - jb_n) \quad n = 0, \pm 1, \pm 2, \ldots$$
the combination of Eqs. (5-2), (5-3), (5-7), and (5-8) yields

$$\alpha_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} y(t) e^{-j2\pi nf_0 t} dt \quad n = 0, \pm 1, \pm 2, \ldots \quad (5-12)$$

The expression of the Fourier series in exponential form (5-11) and the complex coefficients in the form (5-12) is normally the preferred approach in analysis.

**Example 5-1**

Determine the Fourier series of the periodic function illustrated in Fig. 5-1.

\[ y(t) \]

\[ \frac{T_0}{2} \quad \frac{T_0}{2} \]

\[ 2 \]

\[ t \]

**Figure 5-1.** Periodic triangular waveform.

From (5-12), since \( y(t) \) is an even function then

$$\alpha_n = \begin{cases} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} y(t) \cos(2\pi nf_0 t) dt \\ \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} \left( \frac{2}{T_0} t + \frac{4}{T_0^2} \right) \cos(2\pi nf_0 t) dt + \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \left( \frac{2}{T_0} t - \frac{4}{T_0^2} \right) \cos(2\pi nf_0 t) dt \end{cases} \quad n = 0, 1, 3, 5, \ldots$$

$$= \begin{cases} \frac{4}{\pi^2 T_0 n^2} & n = 1, 3, 5, \ldots \\ \frac{1}{T_0} & n = 0 \end{cases} \quad (5-13)$$

Hence

$$y(t) = \frac{1}{T_0} + \frac{8}{\pi^2 T_0} \left[ \cos(2\pi f_0 t) + \frac{1}{3} \cos(6\pi f_0 t) + \frac{1}{5} \cos(10\pi f_0 t) + \cdots \right]$$

where \( f_0 = 1/T_0 \).
5-2 FOURIER SERIES AS A SPECIAL CASE OF THE FOURIER INTEGRAL

Consider the periodic triangular function illustrated in Fig. 5-2(c). From Ex. 5-1 we know that the Fourier series of this waveform is an infinite set of sinusoids. We will now show that an identical relationship can be obtained from the Fourier integral.

To accomplish the derivation we utilize the convolution theorem (4-11). Note that the periodic triangular waveform (period \( T_0 \)) is simply the convolution of the single triangle shown in Fig. 5-2(a), and the infinite sequence of equidistant impulses illustrated in Fig. 5-2(b). Periodic function \( y(t) \) can then be expressed by

\[
y(t) = h(t) * x(t) \tag{5-15}\]

Fourier transforms of both \( h(t) \) and \( x(t) \) have been determined previously and are illustrated in Figs. 5-2(c) and (d), respectively. From the convolution theorem, the desired Fourier transform is the product of these two frequency functions

\[
Y(f) = H(f)X(f) = H(f) \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta \left( f - \frac{n}{T_0} \right) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} H \left( \frac{n}{T_0} \right) \delta \left( f - \frac{n}{T_0} \right) \tag{5-16}\]

Equations (2-40) and (4-16) were used to develop (5-16).

The Fourier transform of the periodic function is then an infinite set of sinusoids (i.e., an infinite sequence of equidistant impulses) with amplitudes of \( H(n/T_0) \). Recall that the Fourier series of a periodic function is an infinite sum of sinusoids with amplitudes given by \( \alpha_n \) (5-12). But note that since the limit of integration of (5-12) is from \(-T_0/2\) to \(T_0/2\) and since

\[
h(t) = y(t) \quad -\frac{T_0}{2} < t < \frac{T_0}{2} \tag{5-17}\]

the function \( y(t) \) can be replaced by \( h(t) \) and (5-12) rewritten in the form

\[
\alpha_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} h(t) e^{-j2\pi n t / T_0} \, dt
= \frac{1}{T_0} H(nf_0) = \frac{1}{T_0} H \left( \frac{n}{T_0} \right) \tag{5-18}\]

Thus the coefficients as derived by means of the Fourier integral and those of the conventional Fourier series are the same for a periodic function. Also, a comparison of Figs. 5-2(c) and (f) reveals that except for a factor \( 1/T_0 \), the coefficients \( \alpha_n \) of the Fourier series expansion of \( y(t) \) equal the values of the Fourier transform \( H(f) \) evaluated at \( n/T_0 \).
Figure 5-2. Graphical convolution theorem development of the Fourier transform of a periodic triangular waveform.
In summary, we point out again that the key to the preceding development is the incorporation of distribution theory into Fourier integral theory. As will be demonstrated in the discussions to follow, this unifying concept is basic to a thorough understanding of the discrete Fourier transform and hence the fast Fourier transform.

5.3 WAVESWAY SAMPLE SAMPLING

In the preceding chapters we have developed a Fourier transform theory which considers both continuous and impulse functions of time. Based on these developments, it is straightforward to extend the theory to include sampled waveforms which are of particular interest in this book. We have developed sufficient tools to investigate in detail the theoretical as well as the visual interpretations of sampled waveforms.

If the function \( h(t) \) is continuous at \( t = T \), then a sample of \( h(t) \) at time equal to \( T \) is expressed as

\[
\hat{h}(t) = h(t)\delta(t - T) = h(T)\delta(t - T)
\]  

(5-19)

where the product must be interpreted in the sense of distribution theory [Eq. (A-12)]. The impulse which occurs at time \( T \) has the amplitude equal to the function at time \( T \). If \( h(t) \) is continuous at \( t = nT \) for \( n = 0, \pm 1, \pm 2, \ldots \),

\[
\hat{h}(t) = \sum_{n=-\infty}^{\infty} h(nT)\delta(t - nT)
\]  

(5-20)

is termed the sampled waveform \( h(t) \) with sample interval \( T \). Sampled \( h(t) \) is then an infinite sequence of equidistant impulses, each of whose amplitude is given by the value of \( h(t) \) corresponding to the time of occurrence of the impulse. Figure 5-3 illustrates graphically the sampling concept. Since Eq. (5-20) is the product of the continuous function \( h(t) \) and the sequence of impulses, we can employ the frequency convolution theorem (4-17) to derive the Fourier transform of the sampled waveform. As illustrated in Fig. 5-3, the sampled function [Fig. 5-3(e)] is equal to the product of the waveform \( h(t) \) shown in Fig. 5-3(a) and the sequence of impulses \( \Delta(t) \) illustrated in Fig. 5-3(b). We call \( \Delta(t) \) the sampling function; the notation \( \Delta(t) \) will always imply an infinite sequence of impulses separated by \( T \). The Fourier transforms of \( h(t) \) and \( \Delta(t) \) are shown in Fig. 5-3(c) and (d), respectively. Note that the Fourier transform of the sampling function \( \Delta(t) \) is \( \Delta(f) \); this function is termed the frequency sampling function. From the frequency convolution theorem, the desired Fourier transform is the convolution of the frequency functions illustrated in Figs. 5-3(c) and (d). The Fourier transform of the sampled waveform is then a periodic function where one period is equal,
Figure 5-3. Graphical frequency convolution theorem development of the Fourier transform of a sampled waveform.
Figure 5.4. Aliased Fourier transform of a waveform sampled at an insufficient rate.
within a constant, to the Fourier transform of the continuous function \( h(t) \).
This last statement is valid only if the sampling interval \( T \) is sufficiently small.

If \( T \) is chosen too large, the results illustrated in Fig. 5-4 are obtained. Note that as the sample interval \( T \) is increased [Figs. 5-3(b) and 5-4(b)], the equidistant impulses of \( \Delta(f) \) become more closely spaced [Figs. 5-3(d) and 5-4(d)]. Because of the decreased spacing of the frequency impulses, their convolution with the frequency function \( H(f) \) [Fig. 5-4(c)] results in the overlapping waveform illustrated in Fig. 5-4(f). This distortion of the desired Fourier transform of a sampled function is known as aliasing. As described, aliasing occurs because the time function was not sampled at a sufficiently high rate, i.e., the sample interval \( T \) is too large. It is then natural to pose the question, “How does one guarantee himself that the Fourier transform of a sampled function is not aliased?” An examination of Figs. 5-4(c) and (d) points up the fact that convolution overlap will occur until the separation of the impulses of \( \Delta(f) \) is increased to \( 1/T = 2f_s \), where \( f_s \) is the highest frequency component of the Fourier transform of the continuous function \( h(t) \). That is, if the sample interval \( T \) is chosen equal to one-half the reciprocal of the highest frequency component, aliasing will not occur. This is an extremely important concept in many fields of scientific application; the reason lies in the fact that we need only retain samples of the continuous waveform to determine a replica of the continuous Fourier transform. Furthermore, if a waveform is sampled such that aliasing does not occur, these samples can be appropriately combined to reconstruct identically the continuous waveform. This is merely a statement of the sampling theorem which we will now investigate.

5.4 SAMPLING THEOREM

The sampling theorem states that if the Fourier transform of a function \( h(t) \) is zero for all frequencies greater than a certain frequency \( f_s \), then the continuous function \( h(t) \) can be uniquely determined from a knowledge of its sampled values,

\[
\hat{h}(f) = h(nT) \sum_{n} \delta(t - nT)
\]

where \( T = \frac{1}{2f_s} \).

In particular, \( h(t) \) is given by

\[
h(t) = T \sum_{n} h(nT) \frac{\sin 2\pi f_s (t - nT)}{\pi(t - nT)}
\]

Constraints of the theorem are illustrated graphically in Fig. 5-5. First, it is necessary that the Fourier transform of \( h(t) \) be zero for frequencies greater than \( f_s \). As shown in Fig. 5-8(c), the example frequency function is band-limited at the frequency \( f_s \); the term band-limited is a shortened way of saying
Figure 5.5. Fourier transform of a waveform sampled at the Nyquist sampling rate.
that the Fourier transform is zero for $|f| > f_s$. The second constraint is that the sample spacing be chosen as $T = \frac{1}{2f_c}$; that is, the impulse functions of Fig. 5-5(d) are required to be separated by $1/T = 2f_s$. This spacing insures that when $\Delta(f)$ and $H(f)$ are convolved there will be no aliasing. Alternately, the functions $H(f)$ and $H(f) \ast \Delta(f)$ as illustrated in Figs. 5-5(c) and (f), respectively, are equal in the interval $|f| < f_s$, within the scaling constant $T$. If $T < \frac{1}{2f_c}$, then aliasing will result; if $T > \frac{1}{2f_c}$, the theorem still holds.

The requirement that $T = \frac{1}{2f_c}$ is simply the maximum spacing between samples for which the theorem holds. Frequency $1/T = 2f_s$ is known as the Nyquist sampling rate. Given that these two constraints are true, the theorem states that $h(t)$ [Fig. 5-5(a)] can be reconstructed from a knowledge of the impulses illustrated in Fig. 5-5(e).

To construct a proof of the sampling theorem, recall from the discussion on constraints of the theorem that the Fourier transform of the sampled function is identical, within the constant $T$, to the Fourier transform of the unsampled function, in the frequency range $-f_s \leq f \leq f_s$. From Fig. 5-5(f), the Fourier transform of the sampled time function is given by $H(f) \ast \Delta(f)$. Hence, as illustrated in Figs. 5-6(a), (b), and (e), the multiplication of a rectangular frequency function of amplitude $T$ with the Fourier transform of the sampled waveform is the Fourier transform $H(f)$;

$$H(f) = [H(f) \ast \Delta(f)]Q(f)$$

(5-23)

The inverse Fourier transform of $H(f)$ is the original waveform $h(t)$ as shown in Fig. 5-6(f). But from the convolution theorem, $h(t)$ is equal to the convolution of the inverse Fourier transforms of $H(f) \ast \Delta(f)$ and of the rectangular frequency function. Hence $h(t)$ is given by the convolution of $h(t)$ $\Delta(t)$ [Fig. 5-6(c)] and $q(t)$ [Fig. 5-6(d)];

$$h(t) = [h(t) \Delta(t)] \ast q(t)$$
$$= \sum_{n=\infty}^{\infty} [h(nT) \delta(t - nT)] \ast q(t)$$
$$= \sum_{n=\infty}^{\infty} h(nT) q(t - nT)$$
$$= T \sum_{n=\infty}^{\infty} h(nT) \sin \left( \frac{2\pi f_s(t - nT)}{T} \right)$$

(5-24)

Function $q(t)$ is given by the Fourier transform pair (2-27). Equation (5-24) is the desired expression for reconstructing $h(t)$ from a knowledge of only the samples of $h(t)$.

We should note carefully that it is possible to reconstruct a sampled waveform perfectly only if the waveform is band-limited. In practice, this condition rarely exists. The solution is to sample at such a rate that aliasing is negligible;
Figure 5-6. Graphical derivation of the sampling theorem.
it may be necessary to filter the signal prior to quantization to insure that there exists, to the extent possible, a band-limited function.

5-5 FREQUENCY SAMPLING THEOREM

Analogous to time domain sampling there exists a sampling theorem in the frequency domain. If a function \( h(t) \) is time-limited, that is

\[
h(t) = 0 \quad |t| > T_e
\]

then its Fourier transform \( H(f) \) can be uniquely determined from equidistant samples of \( H(f) \). In particular, \( H(f) \) is given by

\[
H(f) = \sum_{n=-\infty}^{\infty} H\left(\frac{n}{2T_e}\right) \sin\left[\frac{\pi T_e (f - nT_e)}{f - n2T_e}\right]
\]

The proof is similar to the proof of the time domain sampling theorem.

PROBLEMS

5-1. Find the Fourier series of the periodic waveforms illustrated in Fig. 5-7.

5-2. Determine the Fourier transform of the waveforms illustrated in Fig. 5-7. Compare these results with those of Problem 5-1.

5-3. By using graphical arguments similar to those of Fig. 5-4, determine the Nyquist sampling rate for the time functions whose Fourier transform magnitude functions are illustrated in Fig. 5-8. Are there cases where aliasing can be used to an advantage?

5-4. Graphically justify the bandpass sampling theorem which states that

\[
\text{Critical sampling frequency} = \frac{2f_s}{\text{largest integer not exceeding} \frac{f_s}{f_l - f_u}}
\]

where \( f_s \) and \( f_l \) are the upper and lower cutoff frequencies of the bandpass spectrum.

5-5. Assume that the function \( h(t) = \cos(2\pi t) \) has been sampled at \( t = n/4; n = 0, \pm 1, \pm 2, \ldots \). Sketch \( h(t) \) and indicate the sampled values. Graphically and analytically determine Eq. (5-24) for \( h(t = 7/8) \) where the summation is only over \( n = 2, 3, 4, \) and 5.

5-6. A frequency function (say a filter frequency response) has been determined experimentally in the laboratory and is given by a graphical curve. If it is desired to sample this function for computer storage purposes, what is the minimum frequency sampling interval if the frequency function is to later be totally reconstructed? State all assumptions.
Figure 5-7.
Figure 5.8.
REFERENCES


