APPLYING THE DISCRETE FOURIER TRANSFORM

In Chapter 6 we developed the relationship between the discrete and continuous Fourier transforms. In this chapter we explore the mechanics of applying the discrete Fourier transform to the computation of Fourier transforms and Fourier series. As we will see, the primary concern is one of correctly interpreting these results.

9-1 FOURIER TRANSFORMS

To illustrate the application of the discrete Fourier transform to the computation of Fourier transforms, consider Fig. 9.1. We show in Fig. 9.1(a) the function $e^{-t}$. We wish to compute by means of the discrete Fourier transform an approximation to the Fourier transform of this function.

The first step in applying the discrete transform is to choose the number of samples $N$ and the sample interval $T$. For $N = 32$ and $T = 0.25$ we show the samples of $e^{-t}$ in Fig. 9.1(a). Note that we have defined the sample value at $t = 0$ to be consistent with Eq. (2-43) which states that the value of the function at a discontinuity must be defined to be the mid-value if the inverse Fourier transform is to hold.

We next compute the discrete Fourier transform

$$H\left(\frac{n}{NT}\right) = T \sum_{k=0}^{N-1} [e^{-ikt}] e^{-j2\pi k N} n = 0, 1, \ldots, N - 1 \quad (9.1)$$

Note the scale factor $T$ which is introduced to produce equivalence between the continuous and discrete transforms. These results are shown in Figs. 9.1(b) and (c). In Fig. 9.1(b) we show the real part of Fourier transform as
Figure 9-1. Example of Fourier transform computation via the discrete Fourier transform.
determined in Ex. 2-1 and as computed by (9-1). Note that the discrete transform is symmetrical about $n = N/2$. This follows from the fact that the real part of the transform is even [Eq. (8-11)] and that the results for $n > N/2$ are simply negative frequency results. This latter point is emphasized by plotting a true frequency scale beneath the scale for parameter $n$.

We could have graphed the data of Fig. 9-1(b) in the manner conventionally used to display the continuous Fourier transform; that is, from $-f_0$ to $+f_0$. However, the conventional method of displaying results of the discrete Fourier transform is to graph the results of Eq. (9-1) as a function of the parameter $n$. As long as we remember that those results for $n > N/2$ actually relate to negative frequency results, then we should encounter no interpretation problems.

In Fig. 9-1(c) we illustrate the imaginary part of the Fourier transform (Ex. 2-1) and the discrete transform. As shown, the discrete transform approximates rather poorly the continuous transform for the higher frequencies. To reduce this error it is necessary to decrease the sample interval $T$ and increase $N$.

We note that the imaginary function is odd with respect to $n = N/2$. This follows from Eq. (8-14). Repeating, those results for $n > N/2$ are to be interpreted as negative frequency results.
In summary, applying the discrete Fourier transform to the computation of the Fourier transform only requires that we exercise care in the choice of $T$ and $N$ and interpret the results correctly.

9-2 INVERSE FOURIER TRANSFORM
APPROXIMATION

Assume that we are given the continuous real and imaginary frequency functions considered in the previous discussion and that we wish to determine the corresponding time function by means of the inverse discrete Fourier transform

$$h(kT) = \Delta f \sum_{n} \left[ R(n\Delta f) + jI(n\Delta f) \right] e^{j2\pi nk/N}, \quad k = 0, 1, \ldots, N - 1$$

(9-2)

where $\Delta f$ is the sample interval in frequency. Assume $N = 32$ and $\Delta f = 1/8$.

Since we know that $R(f)$, the real part of the complex frequency function, must be an even function then we fold $R(f)$ about the frequency $f = 2.0$ which corresponds to the sample point $n = N/2$. As shown in Fig. 9-2(a), we simply sample the frequency function up to the point $n = N/2$ and then fold these values about $n = N/2$ to obtain the remaining samples.

In Fig. 9-2(b) we illustrate the method for determining the $N$ samples of the imaginary part of the frequency function. Because the imaginary frequency function is odd, we must not only fold about the sample value $N/2$ but also flip the results. To preserve symmetry, we set the sample at $n = N/2$ to zero.

Computation of (9-2) with the sampled function illustrated in Figs. 9-2(a) and (b) yields the inverse discrete Fourier transform. The result is a complex function whose imaginary part is approximately zero and whose real part is as shown in Fig. 9-2(c). We note that at $k = 0$ the result is approximately equal to the correct mid-value and reasonable agreement is obtained for all but the results for $k$ large. Improvement can be obtained by reducing $\Delta f$ and increasing $N$.

The key to using the discrete inverse Fourier transform for obtaining an approximation to continuous results is to specify the sampled frequency functions correctly. Figures 9-3(a) and (b) illustrate this correct method. One should observe the scale factor $\Delta f$ which was required to give a correct approximation to continuous inverse Fourier transform results.

Equivalent results could have been obtained by using the alternate inversion formula (8-9). To use this relationship, we first conjugate the complex frequency function; that is, the imaginary sampled function illustrated in Fig. 9-2(b) is multiplied by $-1$. Since the resulting time function is
Figure 9-2. Example of inverse Fourier transform computation via the discrete Fourier transform.
real, the final conjugation operation illustrated in Eq. (8-9) can be omitted. Hence we compute

\[ h(kT) = \Delta f \sum_{k=0}^{N-1} [R(n\Delta f) + j(-1) I(n\Delta f)] e^{-j2\pi nk/N} \]  \hspace{1cm} (9-3)

which yields the time function illustrated in Fig. 9-2(c).

### 9-3 FOURIER SERIES HARMONIC ANALYSIS

Application of the discrete Fourier transform to the Fourier harmonic analysis of a waveform [Eq. (5-12)] requires that we compute

\[ H\left(\frac{n}{NT}\right) = \frac{T}{(NT)} \sum_{k=0}^{N-1} h(kT) e^{-j2\pi nk/N} \]  \hspace{1cm} (9-4)

where the divisor \((NT)\) is the time duration or period of the lowest frequency harmonic to be determined. Recall from Chapter 6 that for (9-4) to yield valid results, the \(N\) sample values of \(h(kT)\) must represent exactly one complete period of the periodic function \(h(t)\).

Consider the square wave function illustrated in Fig. 9-3(a). As shown, the function has a period of 8 sec. Thus, if \(N = 32\) then \(T\) must be chosen equal to 0.25 to insure that the 32 samples exactly equal one period.
Figure 9.3. Example of Fourier series harmonic analysis via the discrete Fourier transform.
Figure 9.4. Example of Fourier series harmonic synthesis via the discrete Fourier transform.
Substitution of these sample values into (9-4) yields the results illustrated in Fig. 9-3(b). Vertical solid lines represent the magnitude of the harmonic coefficients as obtained theoretically from Eq. (5-12). As expected, the results are symmetrical about the point \( n = N/2 \). Reasonable results are obtained for the lower order harmonics. Accuracy can be improved for the higher harmonics by decreasing \( T \) and increasing \( N \).

Note that we have introduced significant aliasing evidenced by the fact that the true coefficient values have appreciable magnitude at sample number \( n = N/2 \).

9.4 FOURIER SERIES HARMONIC SYNTHESIS

Harmonic synthesis refers to the procedure of calculating a periodic waveform given the coefficients of the Fourier series [Eq. (5-11)]. To accomplish this analysis using the discrete Fourier transform, we simply compute

\[
h(kT) = \Delta f \sum_{k=-N/2}^{N/2-1} H(n\Delta f) e^{j2\pi k N}
\]  

(9-5)

where \( \Delta f \) must be chosen as an integer multiple of the fundamental harmonic.

To apply (9-5) we must sample the real and imaginary coefficients consistent with the procedures discussed previously. If we consider the previous example, then only the real coefficients must be sampled. As shown in Fig. 9-4(a), these samples are folded about the point \( N/2 \). Note that we have in fact truncated the Fourier series because the sample values nearing \( N/2 \) still have appreciable magnitude.

Computation of (9-5) with the sample values shown in Fig. 9-4(a) yields the synthesized waveform illustrated in Fig. 9-4(b). As shown, the results tend to oscillate about the correct value. These oscillations are due to the well-known Gibbs phenomenon\(^*\) which states that truncation in one domain leads to oscillations in the other domain. To decrease the magnitude of these oscillations, it is necessary to consider more harmonic coefficients; that is, increase \( N \).

Results illustrated in Fig. 9-4(b) could also have been obtained by using the alternate inversion formula (8-9).

9.5 LEAKAGE REDUCTION

In Sec. 6.4 we introduced the effect termed leakage which is inherent in the discrete Fourier transform because of the required time domain truncation. Recall that the truncation of a periodic function at other than a multiple

of the period results in a sharp discontinuity in the time domain, or equivalently results in side-lobes in the frequency domain. These side-lobes are responsible for the additional frequency components which are termed leakage. In this section we will investigate the techniques for computing a discrete Fourier transform with minimum leakage.

For review, let us reconsider the developments illustrated in Fig. 6-3. Recall that time domain truncation of the sampled waveform [Fig. 6-3(d)] results in a frequency domain convolution with a sin $(f)/f$ function. This convolution introduces additional components in the frequency domain because of the side-lobe characteristics of the sin $(f)/f$ function. If the truncation interval is chosen equal to a multiple of the period, the frequency domain sampling function [Fig. 6-3(f)] is coincident with the zeros of the sin $(f)/f$ function. As a result, the side-lobe characteristics of the sin $(f)/f$ function do not alter the discrete Fourier transform results [Fig. 6-4(b)].

To illustrate this point we have computed the discrete Fourier transform of the cosine function illustrated in Fig. 9-5(a). For sample interval $T = 1.0$ and the number of samples $N = 32$, we also show in Fig. 9-5(a) samples of the cosine waveform. Note that the thirty-two samples define exactly four periods of the periodic waveform. In Fig. 9-5(b), we illustrate the magnitude of the discrete Fourier transform of these samples as computed by Eq. (9-4). The results are zero except at the desired frequency.

If the time truncation interval is not chosen equal to a multiple of the period, the side-lobe characteristics of the sin $(f)/f$ frequency function result in a considerable difference in discrete and continuous Fourier transform results (Figs. 6-5 and 6-6). To illustrate this effect, consider the cosine waveform illustrated in Fig. 9-6(a). For $T = 1.0$ and $N = 32$, we also show the sampled waveform in Fig. 9-6(a). Note that the thirty-two points do not define a multiple of the period and as a result a sharp discontinuity has been introduced.

In Fig. 9-6(b) we show the magnitude of the discrete Fourier transform of the samples of Fig. 9-6(a). There exist non-zero frequency components at all discrete frequencies of the discrete transform. As stated previously, the additional frequency components are termed leakage and are a result of the side-lobe characteristics of the sin $(f)/f$ function. To reduce leakage it is necessary to employ a time domain truncation function which has side-lobe characteristics which are of smaller magnitude than those of the sin $(f)/f$ function. The smaller the side-lobes, the less leakage will affect the results of the discrete Fourier transform. Fortunately, there exist truncation functions which exhibit exactly the desired characteristics.

One particularly good truncation function is the Hanning [1] function illustrated in Fig. 9-7(a) and given by

$$x(t) = \frac{1}{2} - \frac{1}{2} \cos \frac{2\pi t}{T_e} \quad 0 \leq t \leq T_e$$  \hspace{1cm} (9-6)
Figure 9-5. Fourier transform of a cosine waveform: truncation interval equal to a multitude of the period.
Figure 9-6. Fourier transform of a cosine waveform: truncation interval not equal to a multiple of the period.
$T_s$ is the truncation interval. The magnitude of the Fourier transform of the Hanning function is given by

$$|X(f)| = \frac{1}{2} Q(f) + \frac{1}{4} \left[ Q(f + \frac{1}{T_s}) + Q(f - \frac{1}{T_s}) \right]$$  \hspace{1cm} (9-7)

where

$$Q(f) = \frac{\sin (\pi T_s f)}{\pi f}$$  \hspace{1cm} (9-8)

As shown in Fig. 9-7(b), this frequency function has very small side-lobes. Other truncation functions have similar properties [1]; however, we choose the Hanning function for its simplicity.
Figure 9-8. Example of applying the Hanning function to reduce leakage in the computation of the discrete Fourier transform.
Because of the low side-lobe characteristics of the Hanning function, we expect that its utilization will significantly reduce the leakage which results from time domain truncation. In Fig. 9-8(a) we show the cosine function of Fig. 9-6(a) multiplied by the Hanning truncation function illustrated in Fig. 9-7(a). Note that the effect of the Hanning function is to reduce the discontinuity which results from the rectangular truncation function.

Figure 9-8(b) illustrates the magnitude of the discrete Fourier transform of the samples of Fig. 9-8(a). As expected, the leakage illustrated in Fig. 9-6(b) has been significantly reduced because of the low side-lobe characteristics of the Hanning truncation function. The non-zero frequency components are considerably broadened or smeared with respect to the desired impulse function. Recall that this is to be expected since the effect of time domain truncation is to convolve the frequency impulse function with the Fourier transform of the truncation function. In general, the more one reduces the leakage, the broader or more smeared the results of the discrete Fourier transform appear. The Hanning function is an acceptable compromise.

PROBLEMS

9-1. Consider $h(t) = e^{-t}$. Sample $h(t)$ with $T = 1$ and $N = 4$. Set $h(0) = 0.5$. Compute Eq. (9-1) for $n = 0, 1, 2, 3$ and sketch the results. What is the correct interpretation of the results for $n = 2$ and $3$? Note the real and imaginary relationships of the discrete frequency function. Do the results of Eq. (9-1) approximate closely continuous Fourier transform results? If not, why not?

9-2. Let $H(f) = R(f) + jI(f)$ where

$$R(f) = \frac{1}{(2\pi f)^2 + 1}$$

$$I(f) = \frac{-2\pi f}{(2\pi f)^2 + 1}$$

For $N = 4$ and $\Delta f = 1/4$, sketch the sampled functions $R(f)$ and $I(f)$ analogously to Figs. 9-2(a) and (b). Compute Eq. (9-2) and sketch the result.

9-3. Given the sampled functions $R(n\Delta f)$ and $I(n\Delta f)$ of Problem 9-2, compute the inverse discrete Fourier transform using the alternate inversion formula. Compare with the inverse transform results of Problem 9-2.

9-4. Consider the function $x(t)$ illustrated in Fig. 5-7(b). Sample $x(t)$ with $N = 6$. What is the sample interval $T$ if we wish to apply the discrete Fourier transform to perform harmonic analysis of the waveform? Compute Eq. (9-4) and sketch the results. Compare with the Fourier series results of Chapter 5. Explain differences in the two results.

PROJECTS

The following projects will require access to a computer.
9-5. Develop a computer program which will compute the discrete Fourier transform of complex time domain waveforms. Use the alternate inversion formula to use this program to also compute the inverse discrete Fourier transform. Refer to this computer program as Program DFT (Discrete Fourier Transform).

9-6. Use Program DFT to validate each of the five example problems of Chapter 9.

9-7. Let \( h(t) = e^{-\alpha t} \). Sample \( h(t) \) with \( T = 0.25 \). Compute the discrete Fourier transform of \( h(kT) \) for \( N = 8, 16, 32, \) and \( 64 \). Compare these results and explain the differences. Repeat for \( T = 0.1 \) and \( T = 1.0 \) and discuss the results.

9-8. Let \( h(t) = \cos(2\pi t) \). Sample \( h(t) \) with \( T = \pi/8 \). Compute the discrete Fourier transform with \( N = 16 \). Compare these results with those of Fig. 6-3(g). Repeat for \( N = 24 \). Compare these results to those of Fig. 6-5(g).

9-9. Consider \( h(t) \) illustrated in Fig. 6-7(a). Let \( T_\phi = 1.0 \). Sample \( h(t) \) with \( T = 0.1 \) and \( N = 10 \). Compute the discrete Fourier transform. Repeat for \( T = 0.2 \) and \( N = 5 \), and for \( T = 0.01 \) and \( N = 100 \). Compare and explain these results.

9-10. Let \( h(t) = te^{-\alpha t}, \ t > 0 \). Compute the discrete Fourier transform. Give rationale for choice of \( T \) and \( N \).

9-11. Let 
\[
\begin{align*}
   h(t) &= 0 & t < 0 \\
   &= \frac{1}{2} & t = 0 \\
   &= 1 & 0 < t < 1 \\
   &= \frac{1}{2} & t = 1 \\
   &= 0 & t > 1 \\
\end{align*}
\]
\( x(t) = h(t) \)
Use the discrete convolution theorem to compute an approximation to \( h(t) \ast x(t) \).

9-12. Consider \( h(t) \) and \( x(t) \) as defined in Problem 9-11. Compute the discrete approximation to the correlation of \( h(t) \) and \( x(t) \). Use the correlation theorem.

9-13. Validate the results of Fig. 7-4 using the discrete convolution theorem.

9-14. Apply the discrete convolution theorem to demonstrate the results of Fig. 7-5.

REFERENCES